International Tables for Crystallography (2006). Vol. D, Section 1.3.1.4, pp. 75–76.

One may note that

$$\begin{array}{lll} S_1 = S_{11}; & S_2 = S_{22}; & S_3 = S_{33}; \\ S_4 = S_{23} + S_{32}; & S_5 = S_{31} + S_{13}; & S_6 = S_{12} + S_{21}. \end{array}$$

The Voigt strain matrix S is of the form

$$\begin{pmatrix} S_1 & S_6 & S_5 \\ S_6 & S_2 & S_4 \\ S_5 & S_4 & S_3 \end{pmatrix}.$$

1.3.1.3.2. *Geometrical interpretation of the coefficients of the strain tensor*

Let us consider an orthonormal system of axes with centre P. We remove nothing from the generality of the following by limiting ourselves to a planar problem and assuming that point P' to which P goes in the deformation lies in the plane x_1Px_2 (Fig. 1.3.1.4). Let us consider two neighbouring points, Q and R, lying on axes Px_1 and Px_2, respectively ($PQ = \mathrm{d}x_1$, $PR = \mathrm{d}x_2$). In the deformation, they go to points Q' and R' defined by

$$\mathbf{QQ}': \begin{cases} \mathrm{d}x'_1 = \mathrm{d}x_1 + (\partial u_1/\partial x_1)\mathrm{d}x_1 \\ \mathrm{d}x'_2 = (\partial u_2/\partial x_1)\mathrm{d}x_1 \\ \mathrm{d}x'_3 = 0 \end{cases}$$

$$\mathbf{RR}': \begin{cases} \mathrm{d}x'_1 = (\partial u_1/\partial x_2)\mathrm{d}x_2 \\ \mathrm{d}x'_2 = \mathrm{d}x_2 + (\partial u_2/\partial x_2)\mathrm{d}x_2 \\ \mathrm{d}x'_3 = 0. \end{cases}$$

As the coefficients $\partial u_i/\partial x_j$ are small, the lengths of $P'Q'$ and $P'R'$ are hardly different from PQ and PR, respectively, and the elongations in the directions Px_1 and Px_2 are

$$\frac{P'Q' - PQ}{PQ} = \frac{\mathrm{d}x'_1 - \mathrm{d}x_1}{\mathrm{d}x_1} = \frac{\partial u_1}{\partial x_1} = S_1$$
$$\frac{P'R' - PR}{PR} = \frac{\mathrm{d}x'_2 - \mathrm{d}x_2}{\mathrm{d}x_2} = \frac{\partial u_2}{\partial x_2} = S_2.$$

The components S_1, S_2, S_3 of the principal diagonal of the Voigt matrix can then be interpreted as the elongations in the three directions Px_1, Px_2 and Px_3. The angles α and β between **PQ** and **P′Q′**, and **PR** and **P′R′**, respectively, are given in the same way by

$$\alpha = \mathrm{d}x'_2/\mathrm{d}x_1 = \partial u_2/\partial x_1; \quad \beta = \mathrm{d}x'_1/dx_2 = \partial u_1/\partial x_2.$$

One sees that the coefficient S_6 of Voigt's matrix is therefore

$$S_6 = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} = \alpha + \beta.$$

The angle $\alpha + \beta$ is equal to the difference between angles $\mathbf{PQ} \wedge \mathbf{PR}$ before deformation and $\mathbf{P}''\mathbf{Q}'' \wedge \mathbf{P}'\mathbf{R}'$ after deformation. The nondiagonal terms of the Voigt matrix therefore represent the shears in the planes parallel to Px_1, Px_2 and Px_3, respectively.

To summarize, if one considers a small cube before deformation, it becomes after deformation an arbitrary parallelepiped; the relative elongations of the three sides are given by the diagonal terms of the strain tensor and the variation of the angles by its nondiagonal terms.

The cubic dilatation (1.3.1.5) is

$$\Delta(B) - 1 = S_1 + S_2 + S_3$$

(taking into account the fact that the coefficients S_{ij} are small).

1.3.1.4. *Particular components of the deformation*

1.3.1.4.1. *Simple elongation*

Matrix M has only one coefficient, e_1, and reduces to (Fig. 1.3.1.5*a*)

$$\begin{pmatrix} e_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The quadric of elongations is reduced to two parallel planes, perpendicular to Ox_1, with the equation $x_1 = \pm 1/\sqrt{|e|}$.

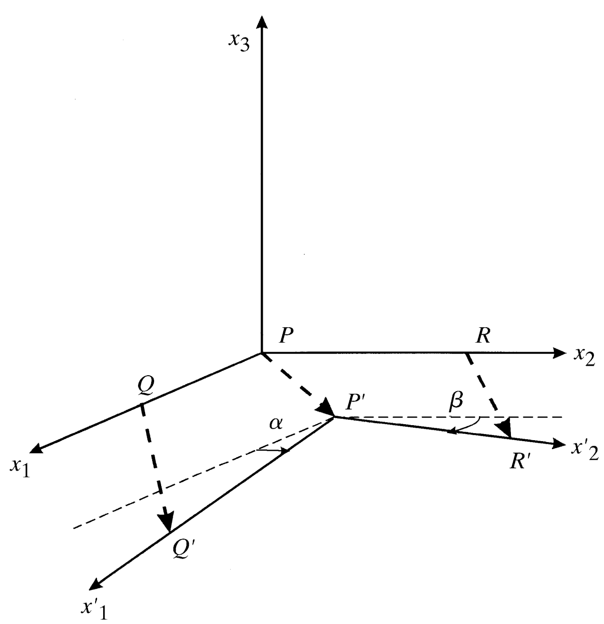

Fig. 1.3.1.4. Geometrical interpretation of the components of the strain tensor. Ox_1, Ox_2, Ox_3: axes before deformation; Ox'_1, Ox'_2, Ox'_3: axes after deformation.

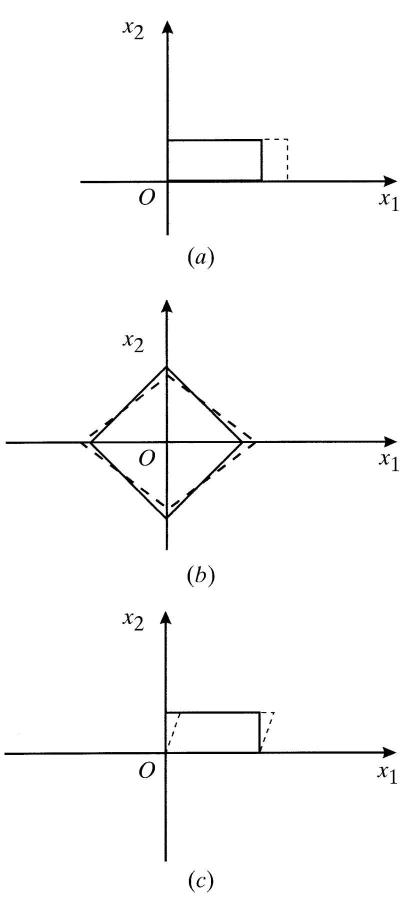

Fig. 1.3.1.5. Special deformations. The state after deformation is represented by a dashed line. (*a*) Simple elongation; (*b*) pure shear; (*c*) simple shear.

1.3.1.4.2. *Pure shear*

This is a pure deformation (without rotation) consisting of the superposition of two simple elongations along two perpendicular directions (Fig. 1.3.1.5*b*) and such that there is no change of volume (the cubic dilatation is zero):

$$\begin{pmatrix} e_1 & 0 & 0 \\ 0 & -e_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The quadric of elongations is a hyperbolic cylinder.

1.3.1.4.3. *Simple shear*

Matrix M_{ij} has one coefficient only, a shear (Fig. 1.3.1.5*c*):

$$\begin{pmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix is not symmetrical, as it contains a component of rotation. Thus we have

$$\left.\begin{aligned} x'_1 &= x_1 + s x_2 \\ x'_2 &= x_2 \\ x'_3 &= x_3. \end{aligned}\right\}$$

One can show that the deformation is a pure shear associated with a rotation around Ox_3.

1.3.2. Stress tensor

1.3.2.1. *General conditions of equilibrium of a solid*

Let us consider a solid C, in movement or not, with a mass distribution defined by a specific mass ρ at each point. There are two types of force that are manifested in the interior of this solid.

(i) *Body forces* (or mass forces), which one can write in the form

$$\mathbf{F}\,\mathrm{d}m = \mathbf{F}\rho\,\mathrm{d}\tau,$$

where $\mathrm{d}\tau$ is a volume element and $\mathrm{d}m$ a mass element. Gravity forces or inertial forces are examples of body forces. One can also envisage body torques (or volume couples), which can arise, for example, from magnetic or electric actions but which will be seen to be neglected in practice.

(ii) *Surface forces or stresses*. Let us imagine a cut in the solid along a surface element $\mathrm{d}\sigma$ of normal $\mathbf{n}$ (Fig. 1.3.2.1). The two lips of the cut that were in equilibrium are now subjected to equal and opposite forces, $\mathbf{R}$ and $\mathbf{R}' = -\mathbf{R}$, which will tend to separate or draw together these two lips. One admits that, when the area element $\mathrm{d}\sigma$ tends towards zero, the ratio $\mathbf{R}/\mathrm{d}\sigma$ tends towards a finite limit, $\mathbf{T}_n$, which is called *stress*. It is a force per unit area of surface, homogeneous to a pressure. It will be considered as positive if it is oriented towards the same side of the surface-area element $\mathrm{d}\sigma$ as the normal $\mathbf{n}$ and negative in the other case. The choice of the orientation of $\mathbf{n}$ is arbitrary. The pressure in a liquid is defined in a similar way but its magnitude is independent of the orientation of $\mathbf{n}$ and its direction is always parallel to $\mathbf{n}$. On the other hand, in a solid the constraint $\mathbf{T}_n$ applied to a surface element is not necessarily normal to the latter and the magnitude and the orientation with respect to the normal change when the orientation of $\mathbf{n}$ changes. A stress is said to be *homogeneous* if the force per unit area acting on a surface element of given orientation and given shape is independent of the position of the element in the body. Other stresses are *inhomogeneous*. Pressure is represented by a scalar, and stress by a rank-two tensor, which will be defined in Section 1.3.2.2.

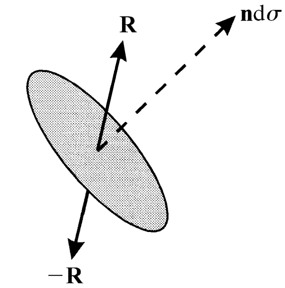

Fig. 1.3.2.1. Definition of stress: it is the limit of $\mathbf{R}\,\mathrm{d}\sigma$ when the surface element $\mathrm{d}\sigma$ tends towards zero. $\mathbf{R}$ and $\mathbf{R}'$ are the forces to which the two lips of the small surface element cut within the medium are subjected.

Now consider a volume V within the solid C and the surface S which surrounds it (Fig. 1.3.2.2). Among the influences that are exterior to V, we distinguish those that are external to the solid C and those that are internal. The first are translated by the body forces, eventually by volume couples. The second are translated by the local contact forces of the part external to V on the internal part; they are represented by a surface density of forces, *i.e.* by the stresses $\mathbf{T}_n$ that depend only on the point Q of the surface S where they are applied and on the orientation of the normal $\mathbf{n}$ of this surface at this point. If two surfaces S and S' are tangents at the same point Q, the same stress acts at the point of contact between them. The equilibrium of the volume V requires:

(i) For the resultant of the applied forces and the inertial forces:

$$\int\!\!\int_S \mathbf{T}_n\,\mathrm{d}\sigma + \int\!\!\int\!\!\int_V \mathbf{F}\rho\,\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\int\!\!\int\!\!\int_V \mathbf{v}\,\mathrm{d}\tau\right\}. \qquad (1.3.2.1)$$

(ii) For the resultant moment:

$$\int\!\!\int_S \mathbf{OQ}\wedge\mathbf{T}_n\,\mathrm{d}\sigma + \int\!\!\int\!\!\int_V \mathbf{OP}\wedge\mathbf{F}\rho\,\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\int\!\!\int\!\!\int_V \mathbf{OP}\wedge\mathbf{v}\,\mathrm{d}\tau\right\}, \qquad (1.3.2.2)$$

where Q is a point on the surface S, P a point in the volume V and $\mathbf{v}$ the velocity of the volume element $\mathrm{d}\tau$.

The equilibrium of the solid C requires that:

(i) there are no stresses applied on its surface and

(ii) the above conditions are satisfied for *any* volume V within the solid C.

1.3.2.2. *Definition of the stress tensor*

Using the condition on the resultant of forces, it is possible to show that the components of the stress $\mathbf{T}_n$ can be determined from the knowledge of the orientation of the normal $\mathbf{n}$ and of the components of a rank-two tensor. Let P be a point situated inside volume V, Px_1, Px_2 and Px_3 three orthonormal axes, and consider a plane of arbitrary orientation that cuts the three axes at Q, R

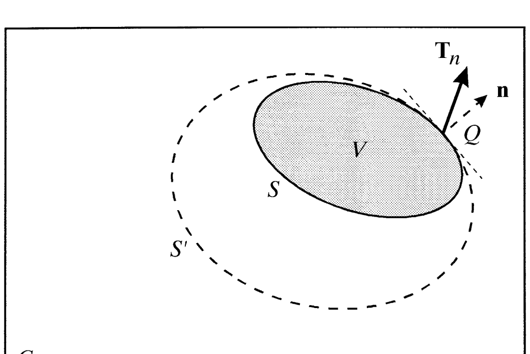

Fig. 1.3.2.2. Stress, $\mathbf{T}_n$, applied to the surface of an internal volume.