International Tables for Crystallography (2006). Vol. D, Section 1.3.2.2, pp. 76–77.

1.3.1.4.2. *Pure shear*

This is a pure deformation (without rotation) consisting of the superposition of two simple elongations along two perpendicular directions (Fig. 1.3.1.5*b*) and such that there is no change of volume (the cubic dilatation is zero):

$$\begin{pmatrix} e_1 & 0 & 0 \\ 0 & -e_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The quadric of elongations is a hyperbolic cylinder.

1.3.1.4.3. *Simple shear*

Matrix M_{ij} has one coefficient only, a shear (Fig. 1.3.1.5*c*):

$$\begin{pmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix is not symmetrical, as it contains a component of rotation. Thus we have

$$\left. \begin{aligned} x'_1 &= x_1 + s x_2 \\ x'_2 &= x_2 \\ x'_3 &= x_3. \end{aligned} \right\}$$

One can show that the deformation is a pure shear associated with a rotation around Ox_3.

1.3.2. Stress tensor

1.3.2.1. *General conditions of equilibrium of a solid*

Let us consider a solid *C*, in movement or not, with a mass distribution defined by a specific mass ρ at each point. There are two types of force that are manifested in the interior of this solid.

(i) *Body forces* (or mass forces), which one can write in the form

$$\mathbf{F}\,\mathrm{d}m = \mathbf{F}\rho\,\mathrm{d}\tau,$$

where $\mathrm{d}\tau$ is a volume element and $\mathrm{d}m$ a mass element. Gravity forces or inertial forces are examples of body forces. One can also envisage body torques (or volume couples), which can arise, for example, from magnetic or electric actions but which will be seen to be neglected in practice.

(ii) *Surface forces or stresses*. Let us imagine a cut in the solid along a surface element $\mathrm{d}\sigma$ of normal **n** (Fig. 1.3.2.1). The two lips of the cut that were in equilibrium are now subjected to equal and opposite forces, **R** and $\mathbf{R}' = -\mathbf{R}$, which will tend to separate or draw together these two lips. One admits that, when the area element $\mathrm{d}\sigma$ tends towards zero, the ratio $\mathbf{R}/\mathrm{d}\sigma$ tends towards a finite limit, $\mathbf{T}_n$, which is called *stress*. It is a force per unit area of surface, homogeneous to a pressure. It will be considered as positive if it is oriented towards the same side of the surface-area element $\mathrm{d}\sigma$ as the normal **n** and negative in the other case. The choice of the orientation of **n** is arbitrary. The pressure in a liquid is defined in a similar way but its magnitude is independent of the orientation of **n** and its direction is always parallel to **n**. On the other hand, in a solid the constraint $\mathbf{T}_n$ applied to a surface element is not necessarily normal to the latter and the magnitude and the orientation with respect to the normal change when the orientation of **n** changes. A stress is said to be *homogeneous* if the force per unit area acting on a surface element of given orientation and given shape is independent of the position of the element in the body. Other stresses are *inhomogeneous*. Pressure is represented by a scalar, and stress by a rank-two tensor, which will be defined in Section 1.3.2.2.

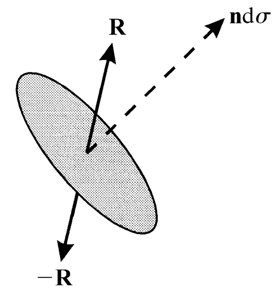

Fig. 1.3.2.1. Definition of stress: it is the limit of **R** $\mathrm{d}\sigma$ when the surface element $\mathrm{d}\sigma$ tends towards zero. **R** and **R**′ are the forces to which the two lips of the small surface element cut within the medium are subjected.

Now consider a volume *V* within the solid *C* and the surface *S* which surrounds it (Fig. 1.3.2.2). Among the influences that are exterior to *V*, we distinguish those that are external to the solid *C* and those that are internal. The first are translated by the body forces, eventually by volume couples. The second are translated by the local contact forces of the part external to *V* on the internal part; they are represented by a surface density of forces, *i.e.* by the stresses $\mathbf{T}_n$ that depend only on the point *Q* of the surface *S* where they are applied and on the orientation of the normal **n** of this surface at this point. If two surfaces *S* and *S*′ are tangents at the same point *Q*, the same stress acts at the point of contact between them. The equilibrium of the volume *V* requires:

(i) For the resultant of the applied forces and the inertial forces:

$$\iint_S \mathbf{T}_n\,\mathrm{d}\sigma + \iiint_V \mathbf{F}\rho\,\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\iiint_V \mathbf{v}\,\mathrm{d}\tau\right\}. \qquad (1.3.2.1)$$

(ii) For the resultant moment:

$$\iint_S \mathbf{OQ}\wedge\mathbf{T}_n\,\mathrm{d}\sigma + \iiint_V \mathbf{OP}\wedge\mathbf{F}\rho\,\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\iiint_V \mathbf{OP}\wedge\mathbf{v}\,\mathrm{d}\tau\right\}, \qquad (1.3.2.2)$$

where *Q* is a point on the surface *S*, *P* a point in the volume *V* and **v** the velocity of the volume element $\mathrm{d}\tau$.

The equilibrium of the solid *C* requires that:

(i) there are no stresses applied on its surface and

(ii) the above conditions are satisfied for *any* volume *V* within the solid *C*.

1.3.2.2. *Definition of the stress tensor*

Using the condition on the resultant of forces, it is possible to show that the components of the stress $\mathbf{T}_n$ can be determined from the knowledge of the orientation of the normal **n** and of the components of a rank-two tensor. Let *P* be a point situated inside volume *V*, Px_1, Px_2 and Px_3 three orthonormal axes, and consider a plane of arbitrary orientation that cuts the three axes at *Q*, *R*

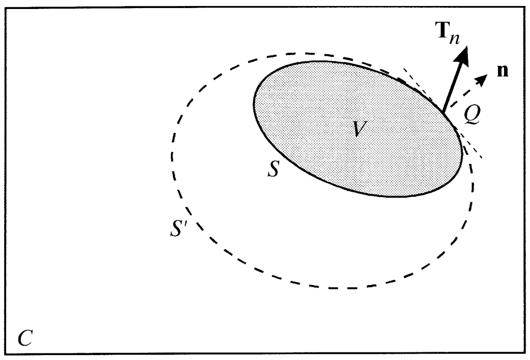

Fig. 1.3.2.2. Stress, $\mathbf{T}_n$, applied to the surface of an internal volume.

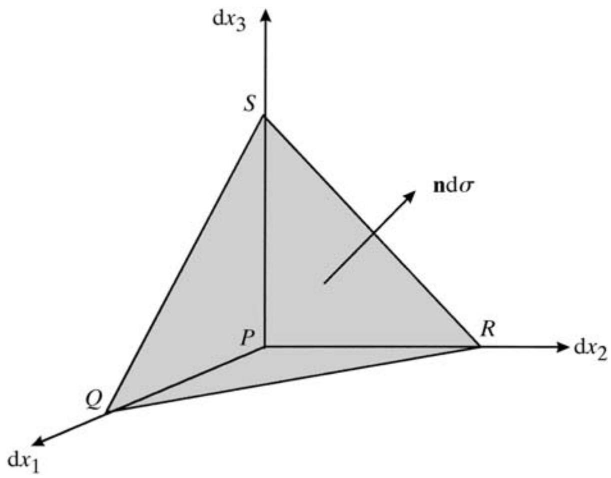

Fig. 1.3.2.3. Equilibrium of a small volume element.

and S, respectively (Fig. 1.3.2.3). The small volume element $PQRS$ is limited by four surfaces to which stresses are applied. The normals to the surfaces PRS, PSQ and PQR will be assumed to be directed towards the interior of the small volume. By contrast, for reasons that will become apparent later, the normal $\mathbf{n}$ applied to the surface QRS will be oriented towards the *exterior*. The corresponding applied forces are thus given in Table 1.3.2.1. The volume $PQRS$ is subjected to five forces: the forces applied to each surface and the resultant of the volume forces and the inertial forces. The equilibrium of the small volume requires that the resultant of these forces be equal to zero and one can write

$$-\mathbf{T}_n \, d\sigma + \mathbf{T}_1 \, d\sigma_1 + \mathbf{T}_2 \, d\sigma_2 + \mathbf{T}_3 \, d\sigma_3 + \mathbf{F}\rho \, d\tau = 0$$

(including the inertial forces in the volume forces).

As long as the surface element $d\sigma$ is finite, however small, it is possible to divide both terms of the equation by it. If one introduces the direction cosines, α_i, the equation becomes

$$-\mathbf{T}_n + \mathbf{T}_1 \, d\alpha_1 + \mathbf{T}_2 \, d\alpha_2 + \mathbf{T}_3 \, d\alpha_3 + \mathbf{F}\rho \, d\tau/d\sigma = 0.$$

When $d\sigma$ tends to zero, the ratio $d\sigma/d\tau$ tends towards zero at the same time and may be neglected. The relation then becomes

$$\mathbf{T}_n = \mathbf{T}_i \alpha^i. \qquad (1.3.2.3)$$

This relation is called the Cauchy relation, which allows the stress $\mathbf{T}_n$ to be expressed as a function of the stresses $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$ that are applied to the three faces perpendicular to the axes, Px_1, Px_2 and Px_3. Let us project this relation onto these three axes:

$$T_{nj} = T_{ij}\alpha_i. \qquad (1.3.2.4)$$

The nine components T_{ij} are, by definition, the components of the stress tensor. In order to check that they are indeed the components of a tensor, it suffices to make the contracted product of each side of (1.3.2.4) by any vector x_j: the left-hand side is a scalar product and the right-hand side a bilinear form. The T_{ij}'s are therefore the components of a tensor. The index to the far left indicates the face to which the stress is applied (normal to the x_1, x_2 or x_3 axis), while the second one indicates on which axis the stress is projected.

Table 1.3.2.1. *Stresses applied to the faces surrounding a volume element*

α_1, α_2 and α_3 are the direction cosines of the normal $\mathbf{n}$ to the small surface QRS.

Face	Area	Applied stress	Applied force
QRS	$d\sigma$	$-\mathbf{T}_n$	$-\mathbf{T}_n \, d\sigma$
PRS	$d\sigma_1 = \alpha_1 \, d\sigma$	$\mathbf{T}_1$	$\mathbf{T}_n \, d\sigma_1$
PSQ	$d\sigma_2 = \alpha_2 \, d\sigma$	$\mathbf{T}_2$	$\mathbf{T}_n \, d\sigma_2$
PQR	$d\sigma_3 = \alpha_3 \, d\sigma$	$\mathbf{T}_3$	$\mathbf{T}_n \, d\sigma_3$

1.3.2.3. *Condition of continuity*

Let us return to equation (1.3.2.1) expressing the equilibrium condition for the resultant of the forces. By replacing $\mathbf{T}_n$ by the expression (1.3.2.4), we get, after projection on the three axes,

$$\int\int_S T_{ij} \, d\sigma_i + \int\int\int_V F_j\rho \, d\tau = 0,$$

where $d\sigma_i = \alpha_i \, d\sigma$ and the inertial forces are included in the volume forces. Applying Green's theorem to the first integral, we have

$$\int\int_S T_{ij} \, d\sigma_i = \int\int\int_V \left[\partial T_{ij}/\partial x_i\right] d\tau.$$

The equilibrium condition now becomes

$$\int\int\int_V \left[\partial T_{ij}/\partial x_i + F_j\rho\right] d\tau = 0.$$

In order that this relation applies to any volume V, the expression under the integral must be equal to zero,

$$\partial T_{ij}/\partial x_i + F_j\rho = 0, \qquad (1.3.2.5)$$

or, if one includes explicitly the inertial forces,

$$\partial T_{ij}/\partial x_i + F_j\rho = -\rho \, \partial^2 x_j/\partial t^2. \qquad (1.3.2.6)$$

This is the condition of continuity or of conservation. It expresses how constraints propagate throughout the solid. This is how the cohesion of the solid is ensured. The resolution of any elastic problem requires solving this equation in terms of the particular boundary conditions of that problem.

1.3.2.4. *Symmetry of the stress tensor*

Let us now consider the equilibrium condition (1.3.2.2) relative to the resultant moment. After projection on the three axes, and using the Cartesian expression (1.1.3.4) of the vectorial products, we obtain

$$\int\int_S \tfrac{1}{2}\varepsilon_{ijk}(x_i T_{lj} - x_j T_{li}) d\sigma_l + \int\int\int_V \left[\tfrac{1}{2}\varepsilon_{ijk}\rho(x_i F_j - x_j F_i) + \Gamma_k\right] d\tau = 0.$$

(including the inertial forces in the volume forces). ε_{ijk} is the permutation tensor. Applying Green's theorem to the first integral and putting the two terms together gives

$$\int\int\int_V \left\{\tfrac{1}{2}\varepsilon_{ijk}\left[\frac{\partial}{\partial x_l}(x_i T_{lj} - x_j T_{li}) + \rho(x_i F_j - x_j F_i)\right] + \Gamma_k\right\} d\tau = 0.$$

In order that this relation applies to any volume V within the solid C, we must have

$$\tfrac{1}{2}\varepsilon_{ijk}\left[\frac{\partial}{\partial x_l}(x_i T_{lj} - x_j T_{li})\right] + \Gamma_k = 0$$

or

$$\tfrac{1}{2}\varepsilon_{ijk}\left[x_i\left(\frac{\partial T_{lj}}{\partial x_l} + F_j\rho\right) - x_j\left(\frac{\partial T_{li}}{\partial x_l} + F_i\rho\right) + T_{ij} - T_{ji}\right] + \Gamma_k = 0.$$

Taking into account the continuity condition (1.3.2.5), this equation reduces to

$$\tfrac{1}{2}\varepsilon_{ijk}\rho[T_{ij} - T_{ji}] + \Gamma_k = 0.$$

A volume couple can occur for instance in the case of a magnetic or an electric field acting on a body that locally possesses magnetic or electric moments. In general, apart from very rare cases, one can ignore these volume couples. One can then deduce that the stress tensor is symmetrical: