International Tables for Crystallography (2006). Vol. D, Section 1.3.2.3, p. 77.

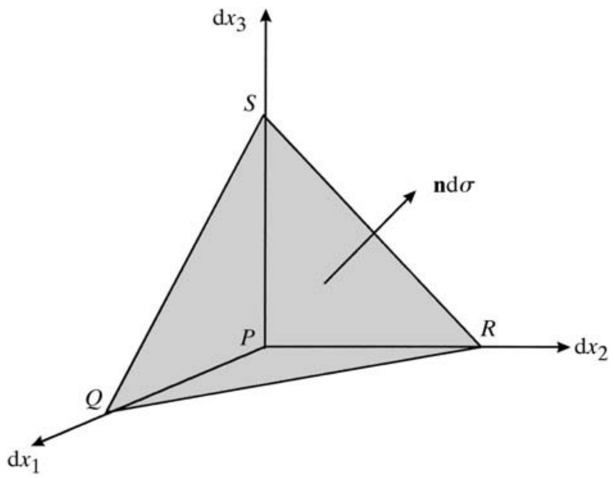

Fig. 1.3.2.3. Equilibrium of a small volume element.

and S, respectively (Fig. 1.3.2.3). The small volume element $PQRS$ is limited by four surfaces to which stresses are applied. The normals to the surfaces PRS, PSQ and PQR will be assumed to be directed towards the interior of the small volume. By contrast, for reasons that will become apparent later, the normal $\mathbf{n}$ applied to the surface QRS will be oriented towards the *exterior*. The corresponding applied forces are thus given in Table 1.3.2.1. The volume $PQRS$ is subjected to five forces: the forces applied to each surface and the resultant of the volume forces and the inertial forces. The equilibrium of the small volume requires that the resultant of these forces be equal to zero and one can write

$$-\mathbf{T}_n \,\mathrm{d}\sigma + \mathbf{T}_1 \,\mathrm{d}\sigma_1 + \mathbf{T}_2 \,\mathrm{d}\sigma_2 + \mathbf{T}_3 \,\mathrm{d}\sigma_3 + \mathbf{F}\rho \,\mathrm{d}\tau = 0$$

(including the inertial forces in the volume forces).

As long as the surface element $\mathrm{d}\sigma$ is finite, however small, it is possible to divide both terms of the equation by it. If one introduces the direction cosines, α_i, the equation becomes

$$-\mathbf{T}_n + \mathbf{T}_1 \,\mathrm{d}\alpha_1 + \mathbf{T}_2 \,\mathrm{d}\alpha_2 + \mathbf{T}_3 \,\mathrm{d}\alpha_3 + \mathbf{F}\rho \,\mathrm{d}\tau/\mathrm{d}\sigma = 0.$$

When $\mathrm{d}\sigma$ tends to zero, the ratio $\mathrm{d}\sigma/\mathrm{d}\tau$ tends towards zero at the same time and may be neglected. The relation then becomes

$$\mathbf{T}_n = \mathbf{T}_i \alpha^i. \qquad (1.3.2.3)$$

This relation is called the Cauchy relation, which allows the stress $\mathbf{T}_n$ to be expressed as a function of the stresses $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$ that are applied to the three faces perpendicular to the axes, Px_1, Px_2 and Px_3. Let us project this relation onto these three axes:

$$T_{nj} = T_{ij}\alpha_i. \qquad (1.3.2.4)$$

The nine components T_{ij} are, by definition, the components of the stress tensor. In order to check that they are indeed the components of a tensor, it suffices to make the contracted product of each side of (1.3.2.4) by any vector x_i: the left-hand side is a scalar product and the right-hand side a bilinear form. The T_{ij}'s are therefore the components of a tensor. The index to the far left indicates the face to which the stress is applied (normal to the x_1, x_2 or x_3 axis), while the second one indicates on which axis the stress is projected.

Table 1.3.2.1. *Stresses applied to the faces surrounding a volume element*

α_1, α_2 and α_3 are the direction cosines of the normal $\mathbf{n}$ to the small surface QRS.

Face	Area	Applied stress	Applied force
QRS	$\mathrm{d}\sigma$	$-\mathbf{T}_n$	$-\mathbf{T}_n \,\mathrm{d}\sigma$
PRS	$\mathrm{d}\sigma_1 = \alpha_1 \,\mathrm{d}\sigma$	$\mathbf{T}_1$	$\mathbf{T}_n \,\mathrm{d}\sigma_1$
PSQ	$\mathrm{d}\sigma_2 = \alpha_2 \,\mathrm{d}\sigma$	$\mathbf{T}_2$	$\mathbf{T}_n \,\mathrm{d}\sigma_2$
PQR	$\mathrm{d}\sigma_3 = \alpha_3 \,\mathrm{d}\sigma$	$\mathbf{T}_3$	$\mathbf{T}_n \,\mathrm{d}\sigma_3$

1.3.2.3. *Condition of continuity*

Let us return to equation (1.3.2.1) expressing the equilibrium condition for the resultant of the forces. By replacing $\mathbf{T}_n$ by the expression (1.3.2.4), we get, after projection on the three axes,

$$\int\!\!\int_S T_{ij} \,\mathrm{d}\sigma_i + \int\!\!\int\!\!\int_V F_j\rho \,\mathrm{d}\tau = 0,$$

where $\mathrm{d}\sigma_i = \alpha_i \,\mathrm{d}\sigma$ and the inertial forces are included in the volume forces. Applying Green's theorem to the first integral, we have

$$\int\!\!\int_S T_{ij} \,\mathrm{d}\sigma_i = \int\!\!\int\!\!\int_V \left[\partial T_{ij}/\partial x_i\right]\mathrm{d}\tau.$$

The equilibrium condition now becomes

$$\int\!\!\int\!\!\int_V \left[\partial T_{ij}/\partial x_i + F_j\rho\right]\mathrm{d}\tau = 0.$$

In order that this relation applies to any volume V, the expression under the integral must be equal to zero,

$$\partial T_{ij}/\partial x_i + F_j\rho = 0, \qquad (1.3.2.5)$$

or, if one includes explicitly the inertial forces,

$$\partial T_{ij}/\partial x_i + F_j\rho = -\rho \,\partial^2 x_j/\partial t^2. \qquad (1.3.2.6)$$

This is the condition of continuity or of conservation. It expresses how constraints propagate throughout the solid. This is how the cohesion of the solid is ensured. The resolution of any elastic problem requires solving this equation in terms of the particular boundary conditions of that problem.

1.3.2.4. *Symmetry of the stress tensor*

Let us now consider the equilibrium condition (1.3.2.2) relative to the resultant moment. After projection on the three axes, and using the Cartesian expression (1.1.3.4) of the vectorial products, we obtain

$$\int\!\!\int_S \tfrac{1}{2}\varepsilon_{ijk}(x_i T_{lj} - x_j T_{li})\mathrm{d}\sigma_l + \int\!\!\int\!\!\int_V \left[\tfrac{1}{2}\varepsilon_{ijk}\rho(x_i F_j - x_j F_i) + \Gamma_k\right]\mathrm{d}\tau = 0.$$

(including the inertial forces in the volume forces). ε_{ijk} is the permutation tensor. Applying Green's theorem to the first integral and putting the two terms together gives

$$\int\!\!\int\!\!\int_V \left\{\tfrac{1}{2}\varepsilon_{ijk}\left[\frac{\partial}{\partial x_l}(x_i T_{lj} - x_j T_{li}) + \rho(x_i F_j - x_j F_i)\right] + \Gamma_k\right\}\mathrm{d}\tau = 0.$$

In order that this relation applies to any volume V within the solid C, we must have

$$\tfrac{1}{2}\varepsilon_{ijk}\left[\frac{\partial}{\partial x_l}(x_i T_{lj} - x_j T_{li})\right] + \Gamma_k = 0$$

or

$$\tfrac{1}{2}\varepsilon_{ijk}\left[x_i\left(\frac{\partial T_{lj}}{\partial x_l} + F_j\rho\right) - x_j\left(\frac{\partial T_{li}}{\partial x_l} + F_i\rho\right) + T_{ij} - T_{ji}\right] + \Gamma_k = 0.$$

Taking into account the continuity condition (1.3.2.5), this equation reduces to

$$\tfrac{1}{2}\varepsilon_{ijk}\rho[T_{ij} - T_{ji}] + \Gamma_k = 0.$$

A volume couple can occur for instance in the case of a magnetic or an electric field acting on a body that locally possesses magnetic or electric moments. In general, apart from very rare cases, one can ignore these volume couples. One can then deduce that the stress tensor is symmetrical:

references