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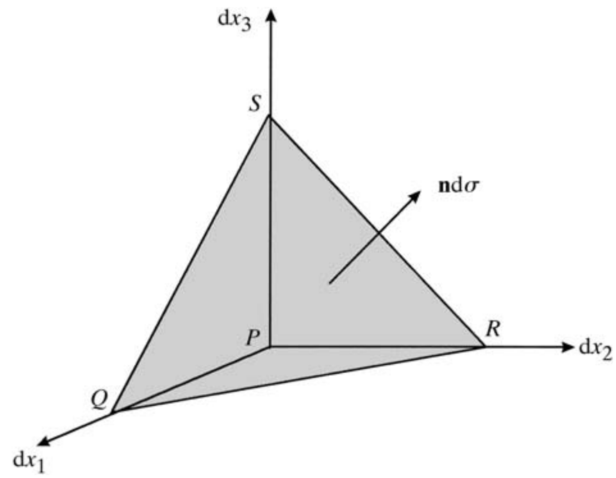

Fig. 1.3.2.3. Equilibrium of a small volume element.

and S, respectively (Fig. 1.3.2.3). The small volume element $PQRS$ is limited by four surfaces to which stresses are applied. The normals to the surfaces PRS, PSQ and PQR will be assumed to be directed towards the interior of the small volume. By contrast, for reasons that will become apparent later, the normal $\mathbf{n}$ applied to the surface QRS will be oriented towards the *exterior*. The corresponding applied forces are thus given in Table 1.3.2.1. The volume $PQRS$ is subjected to five forces: the forces applied to each surface and the resultant of the volume forces and the inertial forces. The equilibrium of the small volume requires that the resultant of these forces be equal to zero and one can write

$$-\mathbf{T}_n\ d\sigma + \mathbf{T}_1\ d\sigma_1 + \mathbf{T}_2\ d\sigma_2 + \mathbf{T}_3\ d\sigma_3 + \mathbf{F}\rho\ d\tau = 0$$

(including the inertial forces in the volume forces).

As long as the surface element $d\sigma$ is finite, however small, it is possible to divide both terms of the equation by it. If one introduces the direction cosines, α_i, the equation becomes

$$-\mathbf{T}_n + \mathbf{T}_1\ d\alpha_1 + \mathbf{T}_2\ d\alpha_2 + \mathbf{T}_3\ d\alpha_3 + \mathbf{F}\rho\ d\tau/d\sigma = 0.$$

When $d\sigma$ tends to zero, the ratio $d\sigma/d\tau$ tends towards zero at the same time and may be neglected. The relation then becomes

$$\mathbf{T}_n = \mathbf{T}_i\alpha^i. \qquad (1.3.2.3)$$

This relation is called the Cauchy relation, which allows the stress $\mathbf{T}_n$ to be expressed as a function of the stresses $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$ that are applied to the three faces perpendicular to the axes, Px_1, Px_2 and Px_3. Let us project this relation onto these three axes:

$$T_{nj} = T_{ij}\alpha_i. \qquad (1.3.2.4)$$

The nine components T_{ij} are, by definition, the components of the stress tensor. In order to check that they are indeed the components of a tensor, it suffices to make the contracted product of each side of (1.3.2.4) by any vector x_i: the left-hand side is a scalar product and the right-hand side a bilinear form. The T_{ij}'s are therefore the components of a tensor. The index to the far left indicates the face to which the stress is applied (normal to the x_1, x_2 or x_3 axis), while the second one indicates on which axis the stress is projected.

Table 1.3.2.1. *Stresses applied to the faces surrounding a volume element*

α_1, α_2 and α_3 are the direction cosines of the normal $\mathbf{n}$ to the small surface QRS.

Face	Area	Applied stress	Applied force
QRS	$d\sigma$	$-\mathbf{T}_n$	$-\mathbf{T}_n\ d\sigma$
PRS	$d\sigma_1 = \alpha_1\ d\sigma$	$\mathbf{T}_1$	$\mathbf{T}_n\ d\sigma_1$
PSQ	$d\sigma_2 = \alpha_2\ d\sigma$	$\mathbf{T}_2$	$\mathbf{T}_n\ d\sigma_2$
PQR	$d\sigma_3 = \alpha_3\ d\sigma$	$\mathbf{T}_3$	$\mathbf{T}_n\ d\sigma_3$

1.3.2.3. *Condition of continuity*

Let us return to equation (1.3.2.1) expressing the equilibrium condition for the resultant of the forces. By replacing $\mathbf{T}_n$ by the expression (1.3.2.4), we get, after projection on the three axes,

$$\int\int_S T_{ij}\ d\sigma_i + \int\int\int_V F_j\rho\ d\tau = 0,$$

where $d\sigma_i = \alpha_i\ d\sigma$ and the inertial forces are included in the volume forces. Applying Green's theorem to the first integral, we have

$$\int\int_S T_{ij}\ d\sigma_i = \int\int\int_V \left[\partial T_{ij}/\partial x_i\right] d\tau.$$

The equilibrium condition now becomes

$$\int\int\int_V \left[\partial T_{ij}/\partial x_i + F_j\rho\right] d\tau = 0.$$

In order that this relation applies to any volume V, the expression under the integral must be equal to zero,

$$\partial T_{ij}/\partial x_i + F_j\rho = 0, \qquad (1.3.2.5)$$

or, if one includes explicitly the inertial forces,

$$\partial T_{ij}/\partial x_i + F_j\rho = -\rho\ \partial^2 x_j/\partial t^2. \qquad (1.3.2.6)$$

This is the condition of continuity or of conservation. It expresses how constraints propagate throughout the solid. This is how the cohesion of the solid is ensured. The resolution of any elastic problem requires solving this equation in terms of the particular boundary conditions of that problem.

1.3.2.4. *Symmetry of the stress tensor*

Let us now consider the equilibrium condition (1.3.2.2) relative to the resultant moment. After projection on the three axes, and using the Cartesian expression (1.1.3.4) of the vectorial products, we obtain

$$\int\int_S \tfrac{1}{2}\varepsilon_{ijk}(x_i T_{lj} - x_j T_{li}) d\sigma_l + \int\int\int_V \left[\tfrac{1}{2}\varepsilon_{ijk}\rho(x_i F_j - x_j F_i) + \Gamma_k\right] d\tau = 0.$$

(including the inertial forces in the volume forces). ε_{ijk} is the permutation tensor. Applying Green's theorem to the first integral and putting the two terms together gives

$$\int\int\int_V \left\{\tfrac{1}{2}\varepsilon_{ijk}\left[\frac{\partial}{\partial x_l}(x_i T_{lj} - x_j T_{li}) + \rho(x_i F_j - x_j F_i)\right] + \Gamma_k\right\} d\tau = 0.$$

In order that this relation applies to any volume V within the solid C, we must have

$$\tfrac{1}{2}\varepsilon_{ijk}\left[\frac{\partial}{\partial x_l}(x_i T_{lj} - x_j T_{li})\right] + \Gamma_k = 0$$

or

$$\tfrac{1}{2}\varepsilon_{ijk}\left[x_i\left(\frac{\partial T_{lj}}{\partial x_l} + F_j\rho\right) - x_j\left(\frac{\partial T_{li}}{\partial x_l} + F_i\rho\right) + T_{ij} - T_{ji}\right] + \Gamma_k = 0.$$

Taking into account the continuity condition (1.3.2.5), this equation reduces to

$$\tfrac{1}{2}\varepsilon_{ijk}\rho[T_{ij} - T_{ji}] + \Gamma_k = 0.$$

A volume couple can occur for instance in the case of a magnetic or an electric field acting on a body that locally possesses magnetic or electric moments. In general, apart from very rare cases, one can ignore these volume couples. One can then deduce that the stress tensor is symmetrical:

$$T_{ij} - T_{ji} = 0.$$

This result can be recovered by applying the relation (1.3.2.2) to a small volume in the form of an elementary parallelepiped, thus illustrating the demonstration using Green's theorem but giving insight into the action of the constraints. Consider a rectangular parallelepiped, of sides $2\Delta x_1$, $2\Delta x_2$ and $2\Delta x_3$, with centre P at the origin of an orthonormal system whose axes Px_1, Px_2 and Px_3 are normal to the sides of the parallelepiped (Fig. 1.3.2.4). In order that the resultant moment with respect to a point be zero, it is necessary that the resultant moments with respect to three axes concurrent in this point are zero. Let us write for instance that the resultant moment with respect to the axis Px_3 is zero. We note that the constraints applied to the faces perpendicular to Px_3 do not give rise to a moment and neither do the components T_{11}, T_{13}, T_{22} and T_{23} of the constraints applied to the faces normal to Px_1 and Px_2 (Fig. 1.3.2.4). The components T_{12} and T_{21} alone have a nonzero moment.

For face 1, the constraint is $T_{12} + (\partial T_{12}/\partial x_1)\Delta x_1$ if T_{12} is the magnitude of the constraint at P. The force applied at face 1 is

$$\left[T_{12} + \frac{\partial T_{12}}{\partial x_1}\Delta x_1\right] 4\Delta x_2 \Delta x_3$$

and its moment is

$$\left[T_{12} + \frac{\partial T_{12}}{\partial x_1}\Delta x_1\right] 4\Delta x_2 \Delta x_3 \Delta x_1.$$

Similarly, the moments of the force on the other faces are

Face 1′ : $-\left[T_{12} + \frac{\partial T_{12}}{\partial x_1}(-\Delta x_1)\right] 4\Delta x_2 \Delta x_3(-\Delta x_1);$

Face 2 : $\left[T_{21} + \frac{\partial T_{21}}{\partial x_2}\Delta x_2)\right] 4\Delta x_1 \Delta x_3 \Delta x_2;$

Face 2′ : $-\left[T_{21} + \frac{\partial T_{21}}{\partial x_2}(-\Delta x_2)\right] 4\Delta x_1 \Delta x_3(-\Delta x_2).$

Noting further that the moments applied to the faces 1 and 1′ are of the same sense, and that those applied to faces 2 and 2′ are of the opposite sense, we can state that the resultant moment is

$$[T_{12} - T_{21}] 8\Delta x_1 \Delta x_2 \Delta x_3 = [T_{12} - T_{21}]\Delta\tau,$$

where $8\Delta x_1 \Delta x_2 \Delta x_3 = \Delta\tau$ is the volume of the small parallelepiped. The resultant moment per unit volume, taking into account the couples in volume, is therefore

$$T_{12} - T_{21} + \Gamma_3.$$

It must equal zero and the relation given above is thus recovered.

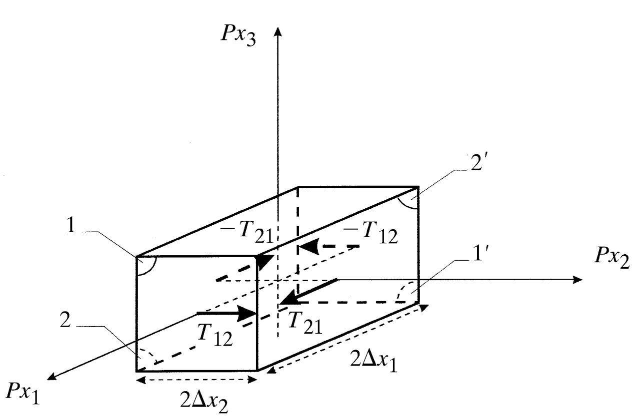

Fig. 1.3.2.4. Symmetry of the stress tensor: the moments of the couples applied to a parallelepiped compensate each other.

1.3.2.5. *Voigt's notation – interpretation of the components of the stress tensor*

1.3.2.5.1. *Voigt's notation, reduced form of the stress tensor*

We shall use frequently the notation due to Voigt (1910) in order to express the components of the stress tensor:

$$\begin{array}{lll} T_1 = T_{11}; & T_2 = T_{22}; & T_3 = T_{33}; \\ T_4 = T_{23} = T_{32}; & T_5 = T_{31} = T_{13}; & T_6 = T_{12} = T_{21}. \end{array}$$

It should be noted that the conventions are different for the Voigt matrices associated with the stress tensor and with the strain tensor (Section 1.3.1.3.1).

The Voigt matrix associated with the stress tensor is therefore of the form

$$\begin{pmatrix} T_1 & T_6 & T_5 \\ T_6 & T_2 & T_4 \\ T_5 & T_4 & T_3 \end{pmatrix}.$$

1.3.2.5.2. *Interpretation of the components of the stress tensor – special forms of the stress tensor*

(i) *Uniaxial stress*: let us consider a solid shaped like a parallelepiped whose faces are normal to three orthonormal axes (Fig. 1.3.2.5). The terms of the main diagonal of the stress tensor correspond to uniaxial stresses on these faces. If there is a single uniaxial stress, the tensor is of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T_3 \end{pmatrix}.$$

The solid is submitted to two equal and opposite forces, $T_{33}S_3$ and $-T_{33}S_3$, where S_3 is the area of the face of the parallelepiped that is normal to the Ox_3 axis (Fig. 1.3.2.5*a*). The convention used in general is that there is a uniaxial *compression* if $T_3 \leq 0$ and a uniaxial *traction* if $T_3 \geq 0$, but the opposite sign convention is sometimes used, for instance in applications such as piezoelectricity or photoelasticity.

(ii) *Pure shear stress*: the tensor reduces to two equal uniaxial constraints of opposite signs (Fig. 1.3.2.5*b*):

$$\begin{pmatrix} T_1 & 0 & 0 \\ 0 & -T_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(iii) *Hydrostatic pressure*: the tensor reduces to three equal uniaxial stresses of the same sign (it is spherical):

$$\begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix},$$

where p is a positive scalar.

(iv) *Simple shear stress*: the tensor reduces to two equal nondiagonal terms (Fig. 1.3.2.5*c*), for instance $T_{12} = T_{21} = T_6$. T_{12} represents the component parallel to Ox_2 of the stress applied to face 1 and T_{21} represents the component parallel to Ox_1 of the stress applied to face 2. These two stresses generate opposite couples that compensate each other. It is important to note that it is impossible to have one nondiagonal term only: its effect would be a couple of rotation of the solid and not a deformation.

1.3.2.6. *Boundary conditions*

If the surface of the solid C is free from all exterior action and is in equilibrium, the stress field T_{ij} inside C is zero at the surface. If C is subjected from the outside to a distribution of stresses T_n