International Tables for Crystallography (2006). Vol. D, Section 1.3.2.6, pp. 78–79.

$$T_{ij} - T_{ji} = 0.$$

This result can be recovered by applying the relation (1.3.2.2) to a small volume in the form of an elementary parallelepiped, thus illustrating the demonstration using Green's theorem but giving insight into the action of the constraints. Consider a rectangular parallelepiped, of sides $2\Delta x_1$, $2\Delta x_2$ and $2\Delta x_3$, with centre P at the origin of an orthonormal system whose axes Px_1, Px_2 and Px_3 are normal to the sides of the parallelepiped (Fig. 1.3.2.4). In order that the resultant moment with respect to a point be zero, it is necessary that the resultant moments with respect to three axes concurrent in this point are zero. Let us write for instance that the resultant moment with respect to the axis Px_3 is zero. We note that the constraints applied to the faces perpendicular to Px_3 do not give rise to a moment and neither do the components T_{11}, T_{13}, T_{22} and T_{23} of the constraints applied to the faces normal to Px_1 and Px_2 (Fig. 1.3.2.4). The components T_{12} and T_{21} alone have a nonzero moment.

For face 1, the constraint is $T_{12} + (\partial T_{12}/\partial x_1)\Delta x_1$ if T_{12} is the magnitude of the constraint at P. The force applied at face 1 is

$$\left[T_{12} + \frac{\partial T_{12}}{\partial x_1}\Delta x_1\right] 4\Delta x_2 \Delta x_3$$

and its moment is

$$\left[T_{12} + \frac{\partial T_{12}}{\partial x_1}\Delta x_1\right] 4\Delta x_2 \Delta x_3 \Delta x_1.$$

Similarly, the moments of the force on the other faces are

$$\text{Face } 1': \quad -\left[T_{12} + \frac{\partial T_{12}}{\partial x_1}(-\Delta x_1)\right] 4\Delta x_2 \Delta x_3(-\Delta x_1);$$

$$\text{Face } 2: \quad \left[T_{21} + \frac{\partial T_{21}}{\partial x_2}\Delta x_2)\right] 4\Delta x_1 \Delta x_3 \Delta x_2;$$

$$\text{Face } 2': \quad -\left[T_{21} + \frac{\partial T_{21}}{\partial x_2}(-\Delta x_2)\right] 4\Delta x_1 \Delta x_3(-\Delta x_2).$$

Noting further that the moments applied to the faces 1 and 1′ are of the same sense, and that those applied to faces 2 and 2′ are of the opposite sense, we can state that the resultant moment is

$$[T_{12} - T_{21}]8\Delta x_1 \Delta x_2 \Delta x_3 = [T_{12} - T_{21}]\Delta\tau,$$

where $8\Delta x_1 \Delta x_2 \Delta x_3 = \Delta\tau$ is the volume of the small parallelepiped. The resultant moment per unit volume, taking into account the couples in volume, is therefore

$$T_{12} - T_{21} + \Gamma_3.$$

It must equal zero and the relation given above is thus recovered.

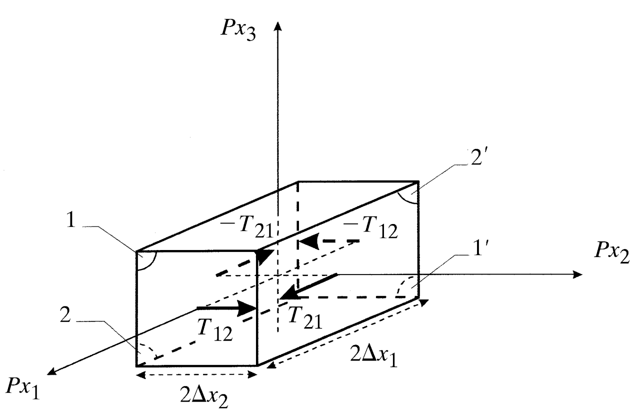

Fig. 1.3.2.4. Symmetry of the stress tensor: the moments of the couples applied to a parallelepiped compensate each other.

1.3.2.5. *Voigt's notation – interpretation of the components of the stress tensor*

1.3.2.5.1. *Voigt's notation, reduced form of the stress tensor*

We shall use frequently the notation due to Voigt (1910) in order to express the components of the stress tensor:

$$\begin{array}{lll} T_1 = T_{11}; & T_2 = T_{22}; & T_3 = T_{33}; \\ T_4 = T_{23} = T_{32}; & T_5 = T_{31} = T_{13}; & T_6 = T_{12} = T_{21}. \end{array}$$

It should be noted that the conventions are different for the Voigt matrices associated with the stress tensor and with the strain tensor (Section 1.3.1.3.1).

The Voigt matrix associated with the stress tensor is therefore of the form

$$\begin{pmatrix} T_1 & T_6 & T_5 \\ T_6 & T_2 & T_4 \\ T_5 & T_4 & T_3 \end{pmatrix}.$$

1.3.2.5.2. *Interpretation of the components of the stress tensor – special forms of the stress tensor*

(i) *Uniaxial stress*: let us consider a solid shaped like a parallelepiped whose faces are normal to three orthonormal axes (Fig. 1.3.2.5). The terms of the main diagonal of the stress tensor correspond to uniaxial stresses on these faces. If there is a single uniaxial stress, the tensor is of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T_3 \end{pmatrix}.$$

The solid is submitted to two equal and opposite forces, $T_{33}S_3$ and $-T_{33}S_3$, where S_3 is the area of the face of the parallelepiped that is normal to the Ox_3 axis (Fig. 1.3.2.5*a*). The convention used in general is that there is a uniaxial *compression* if $T_3 \leq 0$ and a uniaxial *traction* if $T_3 \geq 0$, but the opposite sign convention is sometimes used, for instance in applications such as piezoelectricity or photoelasticity.

(ii) *Pure shear stress*: the tensor reduces to two equal uniaxial constraints of opposite signs (Fig. 1.3.2.5*b*):

$$\begin{pmatrix} T_1 & 0 & 0 \\ 0 & -T_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(iii) *Hydrostatic pressure*: the tensor reduces to three equal uniaxial stresses of the same sign (it is spherical):

$$\begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix},$$

where p is a positive scalar.

(iv) *Simple shear stress*: the tensor reduces to two equal nondiagonal terms (Fig. 1.3.2.5*c*), for instance $T_{12} = T_{21} = T_6$. T_{12} represents the component parallel to Ox_2 of the stress applied to face 1 and T_{21} represents the component parallel to Ox_1 of the stress applied to face 2. These two stresses generate opposite couples that compensate each other. It is important to note that it is impossible to have one nondiagonal term only: its effect would be a couple of rotation of the solid and not a deformation.

1.3.2.6. *Boundary conditions*

If the surface of the solid C is free from all exterior action and is in equilibrium, the stress field T_{ij} inside C is zero at the surface. If C is subjected from the outside to a distribution of stresses T_n

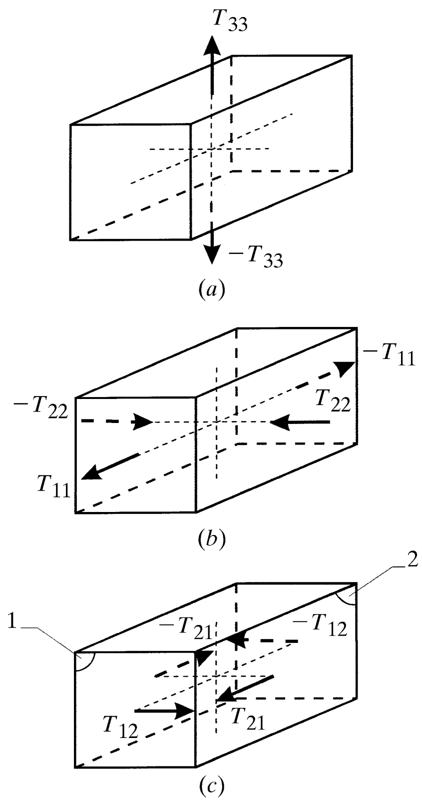

Fig. 1.3.2.5. Special forms of the stress tensor. (*a*) Uniaxial stress: the stress tensor has only one component, T_{33}; (*b*) pure shear stress: $T_{22} = -T_{11}$; (*c*) simple shear stress: $T_{21} = T_{12}$.

(apart from the volume forces mentioned earlier), the stress field inside the solid is such that at each point of the surface

$$T_{nj} = T_{ij}\alpha_i,$$

where the α_j's are the direction cosines of the normal to the surface at the point under consideration.

1.3.2.7. *Local properties of the stress tensor*

(i) *Normal stress and shearing stress*: let us consider a surface area element dσ within the solid, the normal **n** to this element and the stress $\mathbf{T}_n$ that is applied to it (Fig. 1.3.2.6).

The *normal stress*, $\boldsymbol{\nu}$, is, by definition, the component of $\mathbf{T}_n$ on **n**,

$$\boldsymbol{\nu} = \mathbf{n}(\mathbf{T}_n \cdot \mathbf{n})$$

and the *shearing stress*, $\boldsymbol{\tau}$, is the projection of $\mathbf{T}_n$ on the surface area element,

$$\boldsymbol{\tau} = \mathbf{n} \wedge (\mathbf{T}_n \wedge \mathbf{n}) = \mathbf{T}_n - \boldsymbol{\nu}.$$

(ii) *The stress quadric*: let us consider the bilinear form attached to the stress tensor:

$$f(\mathbf{y}) = T_{ij} y_i y_j.$$

The quadric represented by

$$f(\mathbf{y}) = \varepsilon$$

is called the stress quadric, where $\varepsilon = \pm 1$. It may be an ellipsoid or a hyperboloid. Referred to the principal axes, and using Voigt's notation, its equation is

$$y_i^2 T_i = \varepsilon.$$

To every direction **n** of the medium, let us associate the radius vector **y** of the quadric (Fig. 1.3.2.7) through the relation

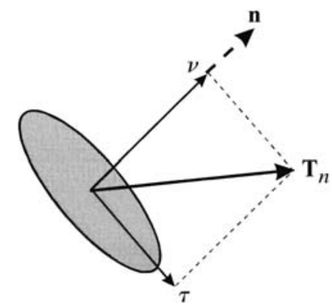

Fig. 1.3.2.6. Normal ($\boldsymbol{\nu}$) and shearing ($\boldsymbol{\tau}$) stress.

$$\mathbf{n} = k\mathbf{y}.$$

The stress applied to a small surface element dσ normal to **n**, $\mathbf{T}_n$, is

$$\mathbf{T}_n = k\nabla(f)$$

and the normal stress, ν, is

$$\nu = \alpha_i T_i = 1/y^2,$$

where the α_i's are the direction cosines of **n**.

(iii) *Principal normal stresses*: the stress tensor is symmetrical and has therefore real eigenvectors. If we represent the tensor with reference to a system of axes parallel to its eigenvectors, it is put in the form

$$\begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix}.$$

T_1, T_2 and T_3 are the principal normal stresses. The mean normal stress, T, is defined by the relation

$$T = (T_1 + T_2 + T_3)/3$$

and is an invariant of the stress tensor.

1.3.2.8. *Energy density in a deformed medium*

Consider a medium that is subjected to a stress field T_{ij}. It has sustained a deformation indicated by the deformation tensor S. During this deformation, the forces of contact have performed work and the medium has accumulated a certain elastic energy W. The knowledge of the energy density thus acquired is useful for studying the properties of the elastic constants. Let the medium deform from the deformation S_{ij} to the deformation $S_{ij} + \delta S_{ij}$ under the influence of the stress field and let us evaluate the work of each component of the effort. Consider a small elementary rectangular parallelepiped of sides $2\Delta x_1$, $2\Delta x_2$, $2\Delta x_3$ (Fig. 1.3.2.8). We shall limit our calculation to the components T_{11} and T_{12}, which are applied to the faces 1 and 1′, respectively.

In the deformation δS, the point P goes to the point P', defined by

$$\mathbf{PP}' = \mathbf{u}(\mathbf{r}).$$

A neighbouring point Q goes to Q' such that (Fig. 1.3.1.1)

$$\mathbf{PQ} = \Delta\mathbf{r}; \quad \mathbf{P}'\mathbf{Q}' = \delta\mathbf{r}'.$$

The coordinates of $\delta\mathbf{r}'$ are given by

$$\delta x_i' = \delta\Delta x_i + \delta S_{ij}\delta x_j.$$

Of sole importance is the relative displacement of Q with respect to P and the displacement that must be taken into account in calculating the forces applied at Q. The coordinates of the relative displacement are