

## 1.3. ELASTIC PROPERTIES

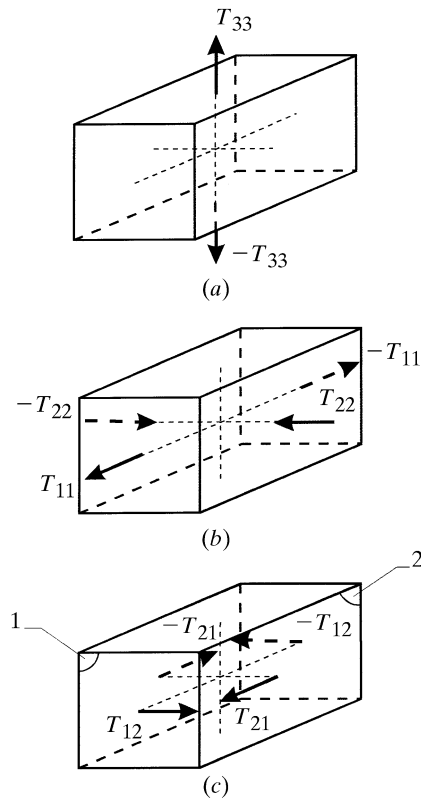


Fig. 1.3.2.5. Special forms of the stress tensor. (a) Uniaxial stress: the stress tensor has only one component,  $T_{33}$ ; (b) pure shear stress:  $T_{22} = -T_{11}$ ; (c) simple shear stress:  $T_{21} = T_{12}$ .

(apart from the volume forces mentioned earlier), the stress field inside the solid is such that at each point of the surface

$$T_{nj} = T_{ij}\alpha_i,$$

where the  $\alpha_j$ 's are the direction cosines of the normal to the surface at the point under consideration.

### 1.3.2.7. Local properties of the stress tensor

(i) *Normal stress and shearing stress*: let us consider a surface area element  $d\sigma$  within the solid, the normal  $\mathbf{n}$  to this element and the stress  $\mathbf{T}_n$  that is applied to it (Fig. 1.3.2.6).

The *normal stress*,  $\nu$ , is, by definition, the component of  $\mathbf{T}_n$  on  $\mathbf{n}$ ,

$$\nu = \mathbf{n}(\mathbf{T}_n \cdot \mathbf{n})$$

and the *shearing stress*,  $\tau$ , is the projection of  $\mathbf{T}_n$  on the surface area element,

$$\boldsymbol{\tau} = \mathbf{n} \wedge (\mathbf{T}_n \wedge \mathbf{n}) = \mathbf{T}_n - \nu \mathbf{n}.$$

(ii) *The stress quadric*: let us consider the bilinear form attached to the stress tensor:

$$f(\mathbf{y}) = T_{ij}y_i y_j.$$

The quadric represented by

$$f(\mathbf{y}) = \varepsilon$$

is called the stress quadric, where  $\varepsilon = \pm 1$ . It may be an ellipsoid or a hyperboloid. Referred to the principal axes, and using Voigt's notation, its equation is

$$y_i^2 T_i = \varepsilon.$$

To every direction  $\mathbf{n}$  of the medium, let us associate the radius vector  $\mathbf{y}$  of the quadric (Fig. 1.3.2.7) through the relation

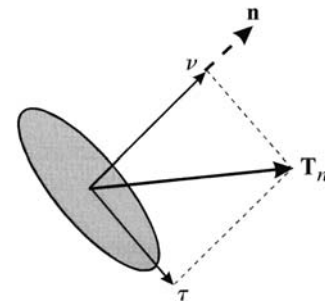


Fig. 1.3.2.6. Normal ( $\nu$ ) and shearing ( $\tau$ ) stress.

$$\mathbf{n} = ky.$$

The stress applied to a small surface element  $d\sigma$  normal to  $\mathbf{n}$ ,  $\mathbf{T}_n$ , is

$$\mathbf{T}_n = k\nabla(f)$$

and the normal stress,  $\nu$ , is

$$\nu = \alpha_i T_i = 1/y^2,$$

where the  $\alpha_i$ 's are the direction cosines of  $\mathbf{n}$ .

(iii) *Principal normal stresses*: the stress tensor is symmetrical and has therefore real eigenvectors. If we represent the tensor with reference to a system of axes parallel to its eigenvectors, it is put in the form

$$\begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix}.$$

$T_1$ ,  $T_2$  and  $T_3$  are the principal normal stresses. The mean normal stress,  $T$ , is defined by the relation

$$T = (T_1 + T_2 + T_3)/3$$

and is an invariant of the stress tensor.

### 1.3.2.8. Energy density in a deformed medium

Consider a medium that is subjected to a stress field  $T_{ij}$ . It has sustained a deformation indicated by the deformation tensor  $S$ . During this deformation, the forces of contact have performed work and the medium has accumulated a certain elastic energy  $W$ . The knowledge of the energy density thus acquired is useful for studying the properties of the elastic constants. Let the medium deform from the deformation  $S_{ij}$  to the deformation  $S_{ij} + \delta S_{ij}$  under the influence of the stress field and let us evaluate the work of each component of the effort. Consider a small elementary rectangular parallelepiped of sides  $2\Delta x_1$ ,  $2\Delta x_2$ ,  $2\Delta x_3$  (Fig. 1.3.2.8). We shall limit our calculation to the components  $T_{11}$  and  $T_{12}$ , which are applied to the faces 1 and 1', respectively.

In the deformation  $\delta S$ , the point  $P$  goes to the point  $P'$ , defined by

$$\mathbf{PP}' = \mathbf{u}(\mathbf{r}).$$

A neighbouring point  $Q$  goes to  $Q'$  such that (Fig. 1.3.1.1)

$$\mathbf{PQ} = \Delta \mathbf{r}; \quad \mathbf{P}'Q' = \delta \mathbf{r}'.$$

The coordinates of  $\delta \mathbf{r}'$  are given by

$$\delta x'_i = \delta \Delta x_i + \delta S_{ij} \delta x_j.$$

Of sole importance is the relative displacement of  $Q$  with respect to  $P$  and the displacement that must be taken into account in calculating the forces applied at  $Q$ . The coordinates of the relative displacement are

# 1. TENSORIAL ASPECTS OF PHYSICAL PROPERTIES

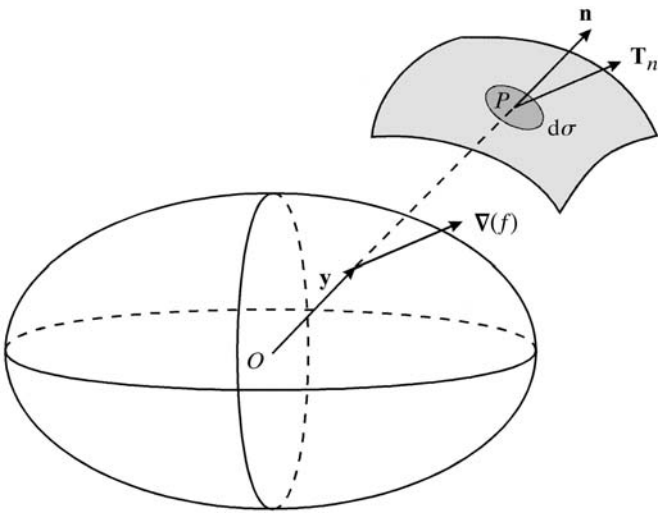


Fig. 1.3.2.7. The stress quadric: application to the determination of the stress applied to a surface element. The surface of the medium is shaded in light grey and a small surface element,  $d\sigma$ , is shaded in medium grey. The stress at  $P$  is proportional to  $\nabla(f)$  at the intersection of  $\mathbf{OP}$  with the stress quadric.

$$\delta x'_i - \delta \Delta x_i = \delta S_{ij} \delta x_j.$$

We shall take as the position of  $Q$  the point of application of the forces at face 1, *i.e.* its centre with coordinates  $\Delta x_1, 0, 0$  (Fig. 1.3.2.8). The area of face 1 is  $4\Delta x_2 \Delta x_3$  and the forces arising from the stresses  $T_{11}$  and  $T_{12}$  are equal to  $4\Delta x_2 \Delta x_3 T_{11}$  and  $4\Delta x_2 \Delta x_3 T_{12}$ , respectively. The relative displacement of  $Q$  parallel to the line of action of  $T_{11}$  is  $\Delta x_1 \delta S_{11}$  and the corresponding displacement along the line of action of  $T_{12}$  is  $\Delta x_1 \delta S_{21}$ . The work of the corresponding forces is therefore

$$\begin{aligned} \text{for } T_{11} : & 4\Delta x_1 \Delta x_2 \Delta x_3 T_{11} \delta S_{11} \\ \text{for } T_{12} : & 4\Delta x_1 \Delta x_2 \Delta x_3 T_{12} \delta S_{21}. \end{aligned}$$

The work of the forces applied to the face  $1'$  is the same ( $T_{11}$ ,  $T_{12}$  and  $x_1$  change sign simultaneously). The works corresponding to the faces 1 and  $1'$  are thus  $T_{11} \delta S_{11} \Delta \tau$  and  $T_{12} \delta S_{21} \Delta \tau$  for the two stresses, respectively. One finds an analogous result for each of the other components of the stress tensor and the total work per unit volume is

$$\delta W = T_{ij} \delta S_{ji}. \quad (1.3.2.7)$$

## 1.3.3. Linear elasticity

### 1.3.3.1. Hooke's law

Let us consider a metallic bar of length  $l_o$  loaded in pure tension (Fig. 1.3.3.1). Under the action of the uniaxial stress  $T = F/A$  ( $F$  applied force,  $A$  area of the section of the bar), the bar elongates and its length becomes  $l = l_o + \Delta l$ . Fig. 1.3.3.2 relates the variations of  $\Delta l$  and of the applied stress  $T$ . The curve representing the traction is very schematic and does not correspond to any real case. The following result, however, is common to all concrete situations:

(i) If  $0 < T < T_o$ , the deformation curve is reversible, *i.e.* if one releases the applied stress the bar resumes its original form. To a first approximation, the curve is linear, so that one can write *Hooke's law*:

$$\frac{\Delta l}{l} = \frac{1}{E} T, \quad (1.3.3.1)$$

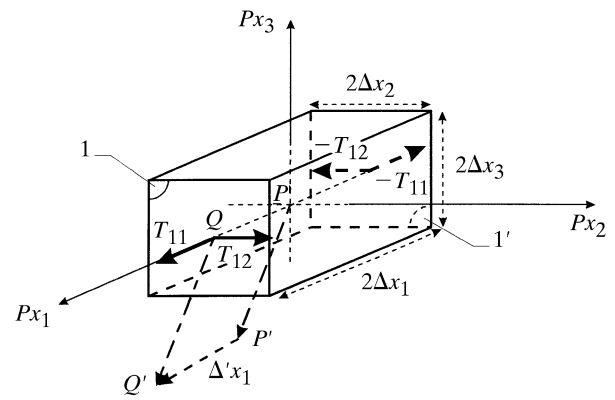


Fig. 1.3.2.8. Determination of the energy density in a deformed medium.  $\mathbf{PP}'$  represents the displacement of the small parallelepiped during the deformation. The thick arrows represent the forces applied to the faces 1 and  $1'$ .

where  $E$  is the elastic stiffness, also called Young's modulus. The physical mechanism at the origin of elasticity is the deformation of the chemical bonds between atoms, ions or molecules in the solid, which act as so many small springs. The reaction of these springs to an applied stress is actually anharmonic and Hooke's law is only an approximation: a Taylor expansion up to the first term. A rigorous treatment of elasticity requires nonlinear phenomena to be taken into account. This is done in Section 1.3.6. The stress below which the strain is recoverable when the stress is removed,  $T_o$ , is called the *elastic limit*.

(ii) If  $T > T_o$ , the deformation curve is no longer reversible. If one releases the applied stress, the bar assumes a permanent deformation. One says that it has undergone a *plastic* deformation. The region of the deformation is ultimately limited by rupture (symbolized by an asterisk on Fig. 1.3.3.2). The plastic deformation is due to the formation and to the movement of lattice defects such as dislocations. The material in its initial state, before the application of a stress, is not free in general from defects and it possesses a complicated history of deformations. The proportionality constant between stresses and deformations in the elastic region depends on the interatomic force constants and is an intrinsic property, very little affected by the presence of defects. By contrast, the limit,  $T_o$ , of the elastic region depends to a large extent on the defects in the material and on its history. It is

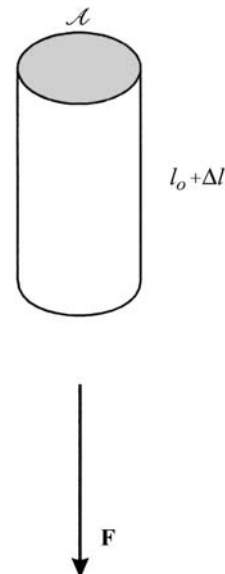


Fig. 1.3.3.1. Bar loaded in pure tension.