The matrix coefficients A_{1i} are the direction cosines of Ox'_1 with respect to the axes Ox_1, Ox_2 and Ox_3. In spherical coordinates, they are given by (Fig. 1.3.3.3)

$$A_{11} = \cos\theta \sin\varphi, \quad A_{12} = \sin\theta \sin\varphi, \quad A_{13} = \cos\varphi,$$

where θ is the angle between Ox'_1 and Ox_1, and φ is the angle between Ox'_1 and Ox_3. Using the reduction of s_{ijkl} for the various crystal classes (Section 1.1.4.9.9), we find, in terms of the reduced two-index components, the following.

(i) *Triclinic system* (groups 1, $\bar{1}$):

$$\begin{aligned} s'_{11} = {} & \big[s_{11}\cos^4\theta + s_{22}\sin^4\theta + (2s_{12}+s_{66})\sin^2 2\theta/4 \\ & + (s_{16}\cos\theta + s_{26}\sin\theta)\sin 2\theta\big]\sin^4\varphi \\ & + 2\big\{\big[(s_{25}+s_{46})\sin\theta + (s_{14}+s_{56})\cos\theta\big]\sin 2\theta/2 \\ & + s_{15}\cos^3\theta + s_{24}\sin^3\theta\big\}\cos\varphi\sin^3\varphi \\ & + \big[(2s_{23}+s_{44})\sin^2\theta + (2s_{13}+s_{55})\cos^2\theta \\ & + (s_{36}+s_{45})\sin 2\theta\big]\sin^2 2\varphi/4 \\ & + 2(s_{35}\cos\theta + s_{34}\sin\theta)\cos^3\varphi\sin\varphi + s_{33}\cos^4\varphi. \end{aligned}$$

(ii) *Monoclinic system* (groups 2, m, $2/m$):

$$\begin{aligned} s'_{11} = {} & \big[s_{11}\cos^4\theta + s_{22}\sin^4\theta + (2s_{12}+s_{66})\sin^2 2\theta/4\big]\sin^4\varphi \\ & + 2\big[(s_{25}+s_{46})\sin^2\theta + s_{15}\cos^2\theta\big]\cos\varphi\sin^3\varphi\cos\theta \\ & + \big[(2s_{23}+s_{44})\sin^2\theta + (2s_{13}+s_{55})\cos^2\theta\big]\sin^2 2\varphi/4 \\ & + 2s_{35}\cos^3\varphi\sin\varphi\cos\theta + s_{33}\cos^4\varphi. \end{aligned}$$

(iii) *Orthorhombic system* (groups 222, $2mm$, mmm):

$$\begin{aligned} s'_{11} = {} & \big[s_{11}\cos^4\theta + s_{22}\sin^4\theta + (2s_{12}+s_{66})\sin^2 2\theta/4\big]\sin^4\varphi \\ & + \big[(2s_{23}+s_{44})\sin^2\theta + (2s_{13}+s_{55})\cos^2\theta\big]\sin^2 2\varphi/4 \\ & + s_{33}\cos^4\varphi. \end{aligned}$$

(iv) *Trigonal system* (groups 3, $\bar{3}$):

$$\begin{aligned} s'_{11} = {} & s_{11}\sin^4\varphi + s_{33}\cos^4\varphi + (2s_{13}+s_{44})\sin^2 2\varphi/4 \\ & + (s_{14}\sin 3\theta - s_{25}\cos 3\theta)\sin 2\varphi\sin^2\varphi. \end{aligned}$$

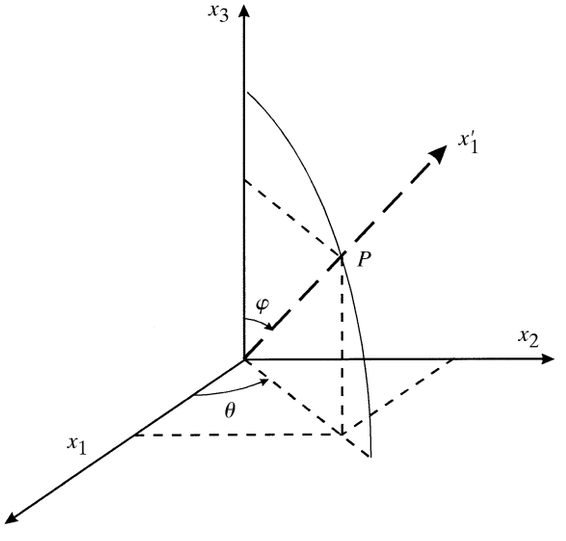

Fig. 1.3.3.3. Spherical coordinates.

(v) *Trigonal system* (groups 32, $3m$, $\bar{3}m$):

$$\begin{aligned} s'_{11} = {} & s_{11}\sin^4\varphi + s_{33}\cos^4\varphi + (2s_{13}+s_{44})\sin^2 2\varphi/4 \\ & + s_{14}\sin 3\theta\sin 2\varphi\sin^2\varphi. \end{aligned}$$

(vi) *Tetragonal system* (groups 4, $\bar{4}$, $4/m$):

$$\begin{aligned} s'_{11} = {} & \big\{s_{11} + \big[s_{66} - 2(s_{11}-s_{12})\big]\sin^2 2\theta/4\big\}\sin^4\varphi + s_{33}\cos^4\varphi \\ & + (2s_{13}+s_{44})\sin^2 2\varphi/4 + s_{16}\sin 4\theta\sin^4\varphi/2. \end{aligned}$$

(vii) *Tetragonal system* (groups 422, $\bar{4}2m$, $4mm$, $4/mmm$):

$$\begin{aligned} s'_{11} = {} & \big\{s_{11} + \big[s_{66} - 2(s_{11}-s_{12})\big]\sin^2 2\theta/4\big\}\sin^4\varphi + s_{33}\cos^4\varphi \\ & + (2s_{13}+s_{44})\sin^2 2\varphi/4. \end{aligned}$$

(viii) *Hexagonal system*:

$$s'_{11} = s_{11}\sin^4\varphi + s_{33}\cos^4\varphi + (2s_{13}+s_{44})\sin^2 2\varphi/4.$$

(ix) *Cubic system*:

$$s'_{11} = s_{11} + \big[s_{44} - 2(s_{11}-s_{12})\big]\sin^2\varphi\big[\cos^2\varphi + \sin^2 2\theta\sin^2\varphi/4\big].$$

This expression reduces to s_{11} if $s_{44} - 2(s_{11}-s_{12}) = 0$ and we retrieve the relation between elastic compliances in an isotropic material (Sections 1.1.4.10.4 and 1.3.3.2.3).

The representation surface of s_{11}, the inverse of Young's modulus, is illustrated in Figure 1.3.3.4 for crystals of different symmetries. As predicted by the Neumann principle, the representation surface is invariant with respect to the symmetry elements of the point group of the crystal but, as stated by the Curie laws, its symmetry can be larger. In the examples of Fig. 1.3.3.4, the symmetry of the surface is the same as that of the point group for sodium chloride (Fig. 1.3.3.4*a*), tungsten (Fig. 1.3.3.4*b*) and aluminium (Fig. 1.3.3.4*c*), which have $m\bar{3}m$ as point group, for tin (Fig. 1.3.3.4*e*, $4/mmm$) and for calcite (Fig. 1.3.3.4*f*, $\bar{3}m$). But in the case of zinc (Fig. 1.3.3.4*d*, $6/mmm$), the surface is of revolution and has a larger symmetry. It is interesting to compare the differences in shapes of the representation surfaces for the three cubic crystals, depending on the value of the anisotropy factor, which is larger than 1 for sodium chloride, smaller than 1 for aluminium and close to 1 for tungsten (see Table 1.3.3.2). In this latter case, the crystal is pseudo-isotropic and the surface is practically a sphere.

1.3.3.5. *Isotropic materials*

The isotropy relation between elastic compliances and elastic stiffnesses is given in Section 1.3.3.2.3. For reasons of symmetry, the directions of the eigenvectors of the stress and strain tensors are necessarily the same in an isotropic medium. If we take these directions as axes, the two tensors are automatically diagonalized and the second relation (1.3.3.7) becomes

$$\begin{aligned} T_1 &= c_{11}S_1 + c_{12}(S_2 + S_3) \\ T_2 &= c_{12}S_1 + c_{11}S_2 + c_{12}S_3 \\ T_3 &= c_{12}(S_1 + S_2) + c_{11}S_3. \end{aligned}$$

These relations can equally well be written in the symmetrical form

$$\begin{aligned} T_1 &= (c_{11} - c_{12})S_1 + c_{12}(S_1 + S_2 + S_3) \\ T_2 &= (c_{11} - c_{12})S_2 + c_{12}(S_1 + S_2 + S_3) \\ T_3 &= (c_{11} - c_{12})S_3 + c_{12}(S_1 + S_2 + S_3). \end{aligned}$$

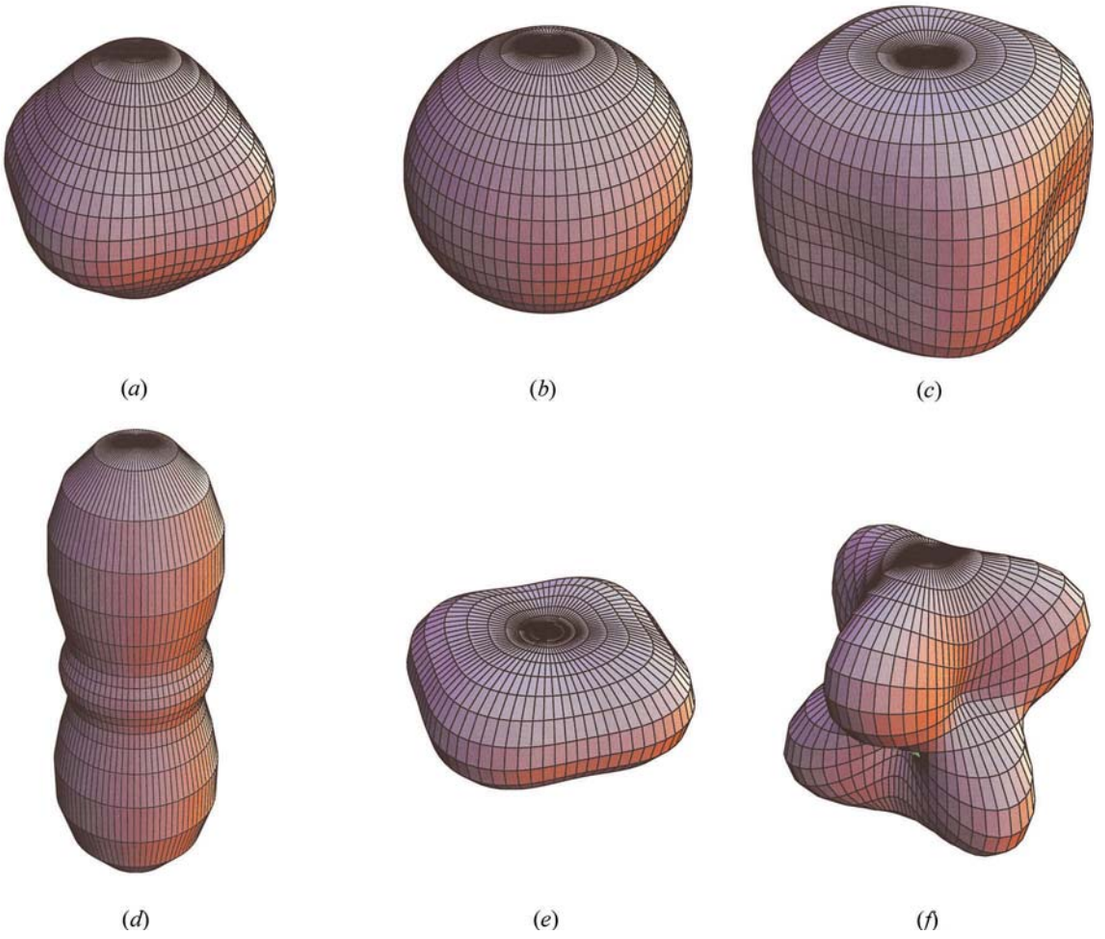

Fig. 1.3.3.4. Representation surface of the inverse of Young's modulus. (*a*) NaCl, cubic, anisotropy factor > 1; (*b*) W, cubic, anisotropy factor $= 1$; (*c*) Al, cubic, anisotropy factor < 1; (*d*) Zn, hexagonal; (*e*) Sn, tetragonal; (*f*) calcite, trigonal.

If one introduces the Lamé constants,

$$\mu = (1/2)(c_{11} - c_{12}) = c_{44}$$
$$\lambda = c_{12},$$

the equations may be written in the form often used in mechanics:

$$\begin{aligned} T_1 &= 2\mu S_1 + \lambda(S_1 + S_2 + S_3) \\ T_2 &= 2\mu S_2 + \lambda(S_1 + S_2 + S_3) \\ T_3 &= 2\mu S_3 + \lambda(S_1 + S_2 + S_3). \end{aligned} \tag{1.3.3.16}$$

Two coefficients suffice to define the elastic properties of an isotropic material, s_{11} and s_{12}, c_{11} and c_{12}, μ and λ, μ and ν, *etc.* Table 1.3.3.3 gives the relations between the more common elastic coefficients.

1.3.3.6. *Equilibrium conditions of elasticity for isotropic media*

We saw in Section 1.3.2.3 that the condition of equilibrium is

$$\partial T_{ij}/\partial x_i + \rho F_j = 0.$$

If we use the relations of elasticity, equation (1.3.3.2), this condition can be rewritten as a condition on the components of the strain tensor:

$$c_{ijkl}\frac{\partial S_{kl}}{\partial x_j} + \rho F_i = 0.$$

Recalling that

$$S_{kl} = \tfrac{1}{2}\left[\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right],$$

the condition becomes a condition on the displacement vector, $\mathbf{u}(\mathbf{r})$:

$$c_{ijkl}\frac{\partial^2}{\partial x_l \partial x_j} + \rho F_i = 0.$$

In an isotropic orthonormal medium, this equation, projected on the axis $0x_1$, can be written with the aid of relations (1.3.3.5) and (1.3.3.9):

$$\begin{aligned} c_{11}\frac{\partial^2 u_1}{(\partial x_1)^2} + c_{12}\left[\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{\partial^2 u_3}{\partial x_1 \partial x_3}\right] \\ + \tfrac{1}{2}(c_{11} - c_{12})\left[\frac{\partial^2 u_1}{(\partial x_2)^2} + \frac{\partial^2 u_3}{\partial x_1 \partial x_3} + \frac{\partial^2 u_1}{(\partial x_3)^2}\right] + \rho F_i \\ = 0. \end{aligned}$$

This equation can finally be rearranged in one of the three following forms with the aid of Table 1.3.3.3.

$$\begin{aligned} \tfrac{1}{2}(c_{11} - c_{12})\Delta\mathbf{u} + \tfrac{1}{2}(c_{11} + c_{12})\nabla(\nabla\mathbf{u}) + \rho\mathbf{F} = 0 \\ \mu\Delta\mathbf{u} + (\mu + \lambda)\nabla(\nabla\mathbf{u}) + \rho\mathbf{F} = 0 \\ \mu\left[\Delta\mathbf{u} + \frac{1}{1 - 2\nu}\nabla(\nabla\mathbf{u})\right] + \rho\mathbf{F} = 0. \end{aligned} \tag{1.3.3.17}$$

references