

## 1. TENSORIAL ASPECTS OF PHYSICAL PROPERTIES

For a solid under finite strain conditions, Hooke's law, valid for infinitesimal deformations, does not hold, and the fundamental definitions for stress and strain must be revisited.

## 1.3.6.2. Lagrangian and Eulerian description

Finite elastic strains may be treated from two different viewpoints using either the *Lagrangian (material)* or the *Eulerian (spatial)* descriptions.

Let us consider a fixed rectangular Cartesian coordinate system with axes  $x_i$  ( $i = 1, 2, 3$ ). Any particular position vector  $\mathbf{r}$  of components  $(x_1, x_2, x_3)$  denotes a point in space. A point that always moves with the material is called a particle or *material point*. Let every particle be identified by its coordinates at some reference time  $t_0$ . These reference coordinates, referred to the same Cartesian system, will be denoted by  $(a_1, a_2, a_3)$  and the corresponding position vector  $\mathbf{a}$ . A particular vector  $\mathbf{a}$  can serve as a name for the particle located at that position at the reference time  $t_0$ .

The vectors  $\mathbf{r}$  and  $\mathbf{a}$  both specify a position in a fixed Cartesian frame of reference. At any time, we associate each  $\mathbf{r}$  with an  $\mathbf{a}$  by the rule that  $\mathbf{r}$  is the present position vector of the particle initially at  $\mathbf{a}$ . This connection between  $\mathbf{r}$  and  $\mathbf{a}$  is written symbolically as

$$\mathbf{r} = \mathbf{r}(t, \mathbf{a}) \quad \text{or} \quad x_i = x_i(t, a_1, a_2, a_3), \quad (1.3.6.1)$$

where

$$\mathbf{a} = \mathbf{r}(t_0, \mathbf{a}) \quad \text{or} \quad a_i = x_i(t_0, a_1, a_2, a_3). \quad (1.3.6.2)$$

The coordinates  $a_i$  that identify the particles are called *material coordinates*. A description that, like (1.3.6.1), uses  $(t, a_1, a_2, a_3)$  as independent variables is called a *material* or *Lagrangian* description.

The converse of (1.3.6.1) and (1.3.6.2) may be written

$$\mathbf{a} = \mathbf{a}(t, \mathbf{r}) \quad \text{or} \quad a_i = a_i(t, x_1, x_2, x_3), \quad (1.3.6.3)$$

where

$$\mathbf{r} = \mathbf{a}(t_0, \mathbf{r}) \quad \text{or} \quad x_i = a_i(t_0, x_1, x_2, x_3). \quad (1.3.6.4)$$

A spatial description or *Eulerian* description uses the independent variables  $(t, x_1, x_2, x_3)$ , the  $x_i$  being called *spatial coordinates*.

Now, for the sake of simplicity, we shall work with the Lagrangian formulation exclusively. For more details see, for instance, Thurston (1964) and Wallace (1970, 1972).

## 1.3.6.3. Strain and stress tensors

The displacement vector from the reference position of a particle to its new position has as components

$$u_i = x_i - a_i. \quad (1.3.6.5)$$

The term strain refers to a change in the relative positions of the material points in a body. Let a final configuration be described in terms of the reference configuration by setting  $t$  equal to a constant in (1.3.6.1). Then  $t$  no longer appears as a variable and (1.3.6.1) can be written

$$x_i = x_i(a_1, a_2, a_3),$$

where the  $a_i$  are the independent variables. It follows that

$$dx_j = \frac{\partial x_j}{\partial a_i} da_i = \left( \frac{\partial u_j}{\partial a_i} + \delta_{ij} \right) da_i. \quad (1.3.6.6)$$

Let now the particle initially at  $(a_1, a_2, a_3)$  move to  $(x_1, x_2, x_3)$ . The square of the initial distance to a neighbouring particle whose initial coordinates were  $a_j + da_j$  is

$$ds^2 = da_j da_j.$$

The square of the final distance to the same neighbouring particle is

$$ds^2 = dx_j dx_j.$$

In a material description, the strain components  $S_{ik}$  are defined by the following equations:

$$dx_j dx_j - da_j da_j = 2S_{ik} da_i da_k. \quad (1.3.6.7)$$

Substituting (1.3.6.6) into (1.3.6.7), it follows that

$$\left( \frac{\partial u_j}{\partial a_i} + \delta_{ji} \right) \left( \frac{\partial u_j}{\partial a_k} + \delta_{jk} \right) da_i da_k - da_j da_j = 2S_{ik} da_i da_k.$$

Hence

$$S_{ik} = \frac{1}{2} \left( \frac{\partial u_k}{\partial a_i} + \frac{\partial u_i}{\partial a_k} + \frac{\partial u_j}{\partial a_i} \frac{\partial u_j}{\partial a_k} \right).$$

If the products and squares of the displacement derivatives are neglected, the strain components reduce to the usual form of 'infinitesimal elasticity' [see equation (1.3.1.8)]:

$$S_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial a_k} + \frac{\partial u_k}{\partial a_i} \right).$$

It is often useful to introduce the Jacobian matrix associated with the transformation  $(\mathbf{a}, \mathbf{x})$ . The components of this matrix are

$$J = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix},$$

where

$$\alpha_{ik} = \frac{\partial x_i}{\partial a_k} = \frac{\partial u_j}{\partial a_k} + \delta_{jk}.$$

From the definition of matrix  $J$ , one has

$$d\mathbf{x} = J d\mathbf{a}$$

and

$$dx^2 - da^2 = (d\mathbf{x})^T d\mathbf{x} - (d\mathbf{a})^T d\mathbf{a} = (d\mathbf{a})^T (J^T J - \delta) d\mathbf{a},$$

where  $(d\mathbf{a})^T$ ,  $(d\mathbf{x})^T$  and  $J^T$  are the transpose matrices of  $d\mathbf{a}$ ,  $d\mathbf{x}$  and  $J$ , respectively, and  $\delta$  is the Kronecker matrix.

The Lagrangian strain matrix  $S$  may then be written symbolically:

$$S = \frac{1}{2} (J^T J - \delta). \quad (1.3.6.8)$$

When finite strains are concerned, we have to distinguish three states of the medium: the natural state, the initial state and the final or present state: The *natural* state is a state free of stress. The *initial* state is deduced from the natural state by a homogeneous strain. The *final* state is deduced from the initial state by an arbitrary strain.

Concerning the stress tensor, as pointed out by Thurston (1964), the stress-deformation relation is complicated in nonlinear elasticity because 'the strain is often referred to a natural unstressed state, whereas the stress  $T_{ij}$  is defined per unit area of the deformed body'. For this reason, the differential of work done by the stress is not equal to the stress components times the differentials of the corresponding strain components. So, following Truesdell & Toupin (1960), we shall introduce a *thermodynamic tension tensor*  $t_{ij}$  defined as the first derivative of the energy with respect to strain. If the internal energy  $U$  per unit