

1.3. ELASTIC PROPERTIES

mass is considered, the *thermodynamic tension* refers to an isentropic process. Then

$$t_{ij}^{\sigma} = \rho_0 \left(\frac{\partial U}{\partial S_{ij}} \right)_{\sigma},$$

where σ is the entropy and ρ_0 the volumic mass in the initial state.

If the Helmholtz free energy F is considered, the thermodynamic tension refers to an isothermal process. Then

$$t_{ij}^{\Theta} = \rho_0 \left(\frac{\partial U}{\partial S_{ij}} \right)_{\Theta},$$

where Θ is the temperature. It will be shown in Section 1.3.7.2 that

$$T_{ij} = (1/J) \alpha_{ik} \alpha_{jl} t_{kl}.$$

1.3.6.4. Second-order and higher-order elastic stiffnesses

Following Brugger (1964), the strain-energy density, or strain energy per unit volume Φ , is assumed to be a polynomial in the strain:

$$\Phi = \Phi_0 + c_{ij} S_{ij} + \frac{1}{2!} c_{ijkl} S_{ij} S_{kl} + \frac{1}{3!} c_{ijklmn} S_{ij} S_{kl} S_{mn}, \quad (1.3.6.9)$$

where $\Phi = \rho_0 U(X, S_{ij})$, $\Phi_0 = \rho_0 U(X, 0)$, X denotes the configuration of the initial state and the S_{ij} 's are the Lagrangian finite strain-tensor components.

If the initial energy and the deformation of the body are both zero, the first two terms in (1.3.6.9) are zero. Note that c_{ij} is a stress and not an intrinsic characteristic of the material. In this expression, the elastic stiffnesses c_{ijkl} and c_{ijklmn} are the second- and third-order stiffnesses, respectively. Since the strain tensor is symmetric, pairs of subscripts can be interchanged [see equation (1.3.3.4)]:

$$\begin{aligned} c_{ijkl} &= c_{jikl} = c_{ijlk} = c_{jilk}, \\ c_{ijklmn} &= c_{jiklmn} = c_{ijlkmn} = c_{jilkmn} = c_{ijklnm} \\ &= c_{jiklnm} = c_{ijlknm} = c_{jilknm}. \end{aligned}$$

More accurately, the isentropic and the isothermal elastic stiffnesses are defined as the n th partial derivatives of the internal energy and the Helmholtz free energy, respectively. For example, the third-order isentropic and isothermal stiffnesses are, respectively,

$$\begin{aligned} c_{ijklmn}^{\sigma} &= \rho_0 \frac{\partial^3 U}{\partial S_{ij} \partial S_{kl} \partial S_{mn}}, \\ c_{ijklmn}^{\Theta} &= \rho_0 \frac{\partial^3 F}{\partial S_{ij} \partial S_{kl} \partial S_{mn}}, \end{aligned}$$

where the internal energy, U , is a function of X , S_{ij} and σ , and the Helmholtz free energy, F , is a function of X , S_{ij} and Θ .

From these definitions, it follows that the Brugger stiffness coefficients depend on the initial state. *When no additional information is given, the initial state is the natural state.*

The third-order stiffnesses form a sixth-rank tensor containing $3^6 = 729$ components, of which 56 are independent for a triclinic crystal and 3 for isotropic materials (the independent components of a sixth-rank tensor can be obtained for any point group using the accompanying software to this volume). The three independent constants for isotropic materials are often taken as c_{123} , c_{144} and c_{456} and denoted respectively by ν_1 , ν_2 , ν_3 , the 'third-order Lamé constants'.

The 'third-order Murnaghan constants' (Murnaghan, 1951), denoted by l , m , n , are given in terms of the Brugger constants by the relations

$$l = \frac{1}{2} c_{112}; \quad m = c_{155}; \quad n = 4c_{456}.$$

Similarly, the fourth-order stiffnesses form an eighth-rank tensor containing $3^8 = 6561$ components, 126 of which are independent for a triclinic crystal and 11 for isotropic materials (the independent components of a sixth-rank tensor can be obtained for any point group using the accompanying software to this volume).

For a solid under finite strain conditions, the definition of the elastic compliance tensor has to be reconsidered. In linear elasticity, the second-order elastic compliances s_{ijkl} were defined through the relations (1.3.3.2):

$$S_{ij} = s_{ijkl} T_{kl} \quad \text{or} \quad s_{ijkl} = \frac{\partial S_{ij}}{\partial T_{kl}},$$

while, in nonlinear elasticity, one has

$$s_{ijkl} = \frac{\partial S_{ij}}{\partial t_{kl}},$$

where

$$t_{kl} = \rho_0 \frac{\partial U}{\partial S_{kl}}.$$

1.3.6.5. Expansion of elastic constants for small initial stress

In most experiments, the initial stress is small compared with the second-order elastic constants (for example, 1 GPa hydrostatic pressure compared with the usual value $c_{ijkl} = 100$ GPa). Consequently, the deformation between the initial (stressed) state and the natural (unstressed) state is small compared with 1. For this reason, it is convenient to expand the elastic constants in the initial state as a power series in the strain about the natural state. *To avoid confusion, we introduce new notations:* \bar{X} now represents the coordinates in the *natural* or unstressed state; X represents the coordinates in the *initial* or homogeneously strained state; $u_i = x_i - X_i$ are the components of displacement. All letters with superscript bar refer to the natural state; for example, \bar{S}_{ij} denotes the Lagrangian strain in the natural state; $\bar{U} = U(\bar{X}, \bar{S}_{ij})$.

Now, in order to relate the properties at X to those at \bar{X} , we need to specify the strain from \bar{X} to X . Let

$$a_{ij} = \frac{\partial X_i}{\partial \bar{X}_j} = \bar{\alpha}_{ij}.$$

Consequently,

$$\frac{\partial \bar{S}_{ij}}{\partial S_{mn}} = a_{mi} a_{nj}.$$

The second-order elastic constants at X can be expressed in terms of the second- and third-order elastic constants at \bar{X} :

$$c_{ijkl} = \rho_0 \frac{\partial^2 U}{\partial S_{ij} \partial S_{kl}} = \rho_0 \frac{\partial^2 \bar{U}}{\partial \bar{S}_{mn} \partial \bar{S}_{pq}} a_{im} a_{jn} a_{kp} a_{lq}$$

or

$$c_{ijkl} = \frac{\rho_0}{\rho_0} \left(\bar{c}_{mnpq} + \bar{c}_{mnpqrs} S_{rs} + \frac{1}{2!} \bar{c}_{mnpqrst} S_{rs} S_{tu} + \dots \right) a_{im} a_{jn} a_{kp} a_{lq}.$$

This expression holds for both isentropic and isothermal elastic constants.