

1.3. ELASTIC PROPERTIES

t_{kl} , the thermodynamic tensor conjugate to the variable S_{kl}/ρ_0 , is generally denoted as the ‘second Piola–Kirchoff stress tensor’.

Using Φ , the strain energy per unit volume, Newton’s law (1.3.7.4) takes the form

$$\rho x_i'' = \frac{\partial}{\partial X_j} \left(\alpha_{jk} \frac{\partial \Phi}{\partial S_{ik}} \right) \quad \text{or} \quad \rho u_i'' = \frac{\partial}{\partial X_j} \left(\alpha_{jk} \frac{\partial \Phi}{\partial S_{ik}} \right)$$

and

$$T_{ij} = \alpha_{jk} \frac{\partial \Phi}{\partial S_{ik}}. \quad (1.3.7.5)$$

1.3.7.3. Wave propagation in a nonlinear elastic medium

As an example, let us consider the case of a plane finite amplitude wave propagating along the x_1 axis. The displacement components in this case become

$$u_1 = u_1(X_1, t); \quad u_2 = u_2(X_1, t); \quad u_3 = u_3(X_1, t).$$

Thus, the Jacobian matrix α_{ij} reduces to

$$J = \begin{pmatrix} \alpha_{11} & 0 & 0 \\ \alpha_{21} & 0 & 0 \\ \alpha_{31} & 0 & 0 \end{pmatrix}.$$

The Lagrangian strain matrix is [equation (1.3.6.8)]

$$S = \frac{1}{2}(J^T J - \delta).$$

The only nonvanishing strain components are, therefore,

$$\begin{aligned} S_{11} &= \frac{1}{2}(\alpha_{11}^2 + \alpha_{21}^2 + \alpha_{31}^2) - 1 \\ &= \frac{\partial u_1}{\partial X_1} + \frac{1}{2} \left[\left(\frac{\partial u_1}{\partial X_1} \right)^2 + \left(\frac{\partial u_2}{\partial X_1} \right)^2 + \left(\frac{\partial u_3}{\partial X_1} \right)^2 \right] \\ S_{12} &= S_{21} = \frac{1}{2} \frac{\partial u_2}{\partial X_1} \\ S_{13} &= S_{31} = \frac{1}{2} \frac{\partial u_3}{\partial X_1} \end{aligned}$$

and the strain invariants reduce to

$$I_1 = S_{11}; \quad I_2 = -(S_{12}S_{21} + S_{13}S_{31}); \quad I_3 = 0.$$

1.3.7.3.1. Isotropic media

In this case, the strain-energy density becomes

$$\Phi = \frac{1}{2}(\lambda + 2\mu)(S_{11})^2 + 2\mu(S_{12}S_{21} + S_{13}S_{31}) + \frac{1}{3}(l + 2m)(S_{11})^3 + 2mS_{11}(S_{12}S_{21} + S_{13}S_{31}). \quad (1.3.7.6)$$

Differentiating (1.3.7.6) with respect to the strains, we get

$$\frac{\partial \Phi}{\partial S_{11}} = (\lambda + 2\mu)S_{11} + (l + 2m)(S_{11})^2 + 2m(S_{12}S_{21} + S_{13}S_{31})$$

$$\frac{\partial \Phi}{\partial S_{12}} = 2\mu S_{21} + 2mS_{11}S_{21}$$

$$\frac{\partial \Phi}{\partial S_{13}} = 2\mu S_{31} + 2mS_{11}S_{31}$$

$$\frac{\partial \Phi}{\partial S_{21}} = 2\mu S_{12} + 2mS_{11}S_{12}$$

$$\frac{\partial \Phi}{\partial S_{31}} = 2\mu S_{13} + 2mS_{11}S_{13}.$$

All the other $\partial \Phi / \partial S_{ij} = 0$.

From (1.3.7.5), we derive the stress components:

$$T_{11} = \alpha_{1k} \frac{\partial \Phi}{\partial S_{1k}}; \quad T_{12} = \alpha_{2k} \frac{\partial \Phi}{\partial S_{1k}}; \quad T_{13} = \alpha_{3k} \frac{\partial \Phi}{\partial S_{1k}};$$

$$T_{21} = \alpha_{1k} \frac{\partial \Phi}{\partial S_{2k}}; \quad T_{22} = \alpha_{2k} \frac{\partial \Phi}{\partial S_{2k}}; \quad T_{23} = \alpha_{3k} \frac{\partial \Phi}{\partial S_{2k}};$$

$$T_{31} = \alpha_{1k} \frac{\partial \Phi}{\partial S_{3k}}; \quad T_{32} = \alpha_{2k} \frac{\partial \Phi}{\partial S_{3k}}; \quad T_{33} = \alpha_{3k} \frac{\partial \Phi}{\partial S_{3k}}.$$

Note that this tensor is not symmetric.

For the particular problem discussed here, the three components of the equation of motion are

$$\rho u_1'' = dT_{11}/dX_1,$$

$$\rho u_2'' = dT_{21}/dX_1,$$

$$\rho u_3'' = dT_{31}/dX_1.$$

If we retain only terms up to the quadratic order in the displacement gradients, we obtain the following equations of motion:

$$\begin{aligned} \rho u_1'' &= (\lambda + 2\mu) \frac{\partial^2 u_1}{\partial X_1^2} + [3(\lambda + 2\mu) + 2(l + 2m)] \frac{\partial u_1}{\partial X_1} \frac{\partial^2 u_1}{\partial X_1^2} \\ &\quad + (\lambda + 2\mu + m) \left[\frac{\partial u_2}{\partial X_1} \frac{\partial^2 u_2}{\partial X_1^2} + \frac{\partial u_3}{\partial X_1} \frac{\partial^2 u_3}{\partial X_1^2} \right] \\ \rho u_2'' &= \mu \frac{\partial^2 u_2}{\partial X_1^2} + (\lambda + 2\mu + m) \left[\frac{\partial u_1}{\partial X_1} \frac{\partial^2 u_2}{\partial X_1^2} + \frac{\partial u_2}{\partial X_1} \frac{\partial^2 u_1}{\partial X_1^2} \right] \\ \rho u_3'' &= \mu \frac{\partial^2 u_3}{\partial X_1^2} + (\lambda + 2\mu + m) \left[\frac{\partial u_1}{\partial X_1} \frac{\partial^2 u_3}{\partial X_1^2} + \frac{\partial u_3}{\partial X_1} \frac{\partial^2 u_1}{\partial X_1^2} \right]. \end{aligned} \quad (1.3.7.7)$$

1.3.7.3.2. Cubic media (most symmetrical groups)

In this case, the strain-energy density becomes

$$\begin{aligned} \Phi &= \frac{1}{2}c_{11}(S_{11})^2 + c_{44}[(S_{12})^2 + (S_{21})^2 + (S_{31})^2 + (S_{13})^2] \\ &\quad + c_{111}(S_{11})^3 + \frac{1}{2}c_{166}S_{11}[(S_{12})^2 + (S_{21})^2 + (S_{31})^2 + (S_{13})^2]. \end{aligned} \quad (1.3.7.8)$$

Differentiating (1.3.7.8) with respect to the strain, one obtains

$$\frac{\partial \Phi}{\partial S_{11}} = c_{11}S_{11} + 3c_{111}(S_{11})^2 + \frac{1}{2}c_{166}[(S_{12})^2 + (S_{21})^2 + (S_{31})^2 + (S_{13})^2]$$

$$\frac{\partial \Phi}{\partial S_{21}} = 2c_{44}S_{21} + c_{166}S_{11}S_{21}$$

$$\frac{\partial \Phi}{\partial S_{31}} = 2c_{44}S_{31} + c_{166}S_{11}S_{31}.$$

All other $\partial \Phi / \partial S_{ij} = 0$. From (1.3.7.5), we derive the stress components:

$$T_{11} = \alpha_{1k} \frac{\partial \Phi}{\partial S_{1k}}$$

$$T_{21} = \alpha_{1k} \frac{\partial \Phi}{\partial S_{2k}}$$

$$T_{31} = \alpha_{1k} \frac{\partial \Phi}{\partial S_{3k}}.$$

In this particular case, the three components of the equation of motion are

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$$\begin{aligned}\rho u_1'' &= dT_{11}/dX_1 \\ \rho u_2'' &= dT_{21}/dX_1 \\ \rho u_3'' &= dT_{31}/dX_1.\end{aligned}$$

If we retain only terms up to the quadratic order in the displacement gradients, we obtain the following equations of motion:

$$\begin{aligned}\rho u_1'' &= c_{11} \frac{\partial^2 u_1}{\partial X_1^2} + [3c_{11} + c_{111}] \frac{\partial u_1}{\partial X_1} \frac{\partial^2 u_1}{\partial X_1^2} \\ &\quad + (c_{11} + c_{166}) \left[\frac{\partial u_2}{\partial X_1} \frac{\partial^2 u_2}{\partial X_1^2} + \frac{\partial u_3}{\partial X_1} \frac{\partial^2 u_3}{\partial X_1^2} \right] \\ \rho u_2'' &= c_{44} \frac{\partial^2 u_2}{\partial X_1^2} + (c_{11} + c_{166}) \left[\frac{\partial u_1}{\partial X_1} \frac{\partial^2 u_2}{\partial X_1^2} + \frac{\partial u_2}{\partial X_1} \frac{\partial^2 u_1}{\partial X_1^2} \right] \\ \rho u_3'' &= c_{44} \frac{\partial^2 u_3}{\partial X_1^2} + (c_{11} + c_{166}) \left[\frac{\partial u_1}{\partial X_1} \frac{\partial^2 u_3}{\partial X_1^2} + \frac{\partial u_3}{\partial X_1} \frac{\partial^2 u_1}{\partial X_1^2} \right],\end{aligned}\tag{1.3.7.9}$$

which are identical to (1.3.7.7) if we put

$$c_{11} = \lambda + 2\mu; \quad c_{44} = \mu; \quad c_{111} = 2(l + 2m); \quad c_{166} = m.$$

1.3.7.4. Harmonic generation

The coordinates in the medium free of stress are denoted either a or \bar{X} . The notation \bar{X} is used when we have to discriminate the natural configuration, \bar{X} , from the initial configuration X . Here, the process that we describe refers to the propagation of an elastic wave in a medium free of stress (natural state) and the coordinates will be denoted a_i .

Let us first examine the case of a pure longitudinal mode, *i.e.*

$$u_1 = u_1(a_1, t); \quad u_2 = u_3 = 0.$$

The equations of motion, (1.3.7.7) and (1.3.7.9), reduce to

$$\rho u_1'' = (\lambda + 2\mu) \frac{\partial^2 u_1}{\partial a_1^2} + [3(\lambda + 2\mu) + 2(l + 2m)] \frac{\partial u_1}{\partial a_1} \frac{\partial^2 u_1}{\partial a_1^2}$$

for an isotropic medium or

$$\rho u_1'' = c_{11} \frac{\partial^2 u_1}{\partial a_1^2} + [3c_{11} + c_{166}] \frac{\partial u_1}{\partial a_1} \frac{\partial^2 u_1}{\partial a_1^2}$$

for a cubic crystal (most symmetrical groups) when a pure longitudinal mode is propagated along [100].

For both cases, we have a one-dimensional problem; (1.3.7.7) and (1.3.7.9) can therefore be written

$$\rho u_1'' = K_2 \frac{\partial^2 u_1}{\partial a_1^2} + [3K_2 + K_3] \frac{\partial u_1}{\partial a_1} \frac{\partial^2 u_1}{\partial a_1^2}.\tag{1.3.7.10}$$

The same equation is also valid when a pure longitudinal mode is propagated along [110] and [111], with the following correspondence:

$$\begin{aligned}[100] \quad K_2 &= c_{11}, \quad K_3 = c_{111} \\ [110] \quad K_2 &= \frac{c_{11} + c_{12} + 2c_{44}}{2}, \quad K_3 = \frac{c_{111} + 3c_{112} + 12c_{166}}{4} \\ [111] \quad K_2 &= \frac{c_{11} + 2c_{12} + 4c_{44}}{3}, \\ K_3 &= \frac{c_{111} + 6c_{112} + 12c_{144} + 24c_{166} + 2c_{123} + 16c_{456}}{9}.\end{aligned}$$

Let us assume that $K_3 \ll K_2$; a perturbation solution to (1.3.7.10) is

$$u = u^0 + u^1,$$

where $u^1 \ll u^0$ with

$$u^0 = A \sin(ka - \omega t)\tag{1.3.7.11}$$

$$u^1 = Ba \sin 2(ka - \omega t) + Ca \cos 2(ka - \omega t).\tag{1.3.7.12}$$

If we substitute the trial solutions into (1.3.7.10), we find after one iteration the following approximate solution:

$$u = A \sin(ka - \omega t) - \frac{(kA)^2(3K_2 + K_3)}{8\rho c^2} a \cos 2(ka - \omega t),$$

which involves second-harmonic generation.

If additional iterations are performed, higher harmonic terms will be obtained. A well known property of the first-order nonlinear equation (1.3.7.10) is that its solutions exhibit discontinuous behaviour at some point in space and time. It can be seen that such a discontinuity would appear at a distance from the origin given by (Breazeale, 1984)

$$L = -2 \frac{(K_2)^2}{3K_2 + K_3} \rho \omega u_0',$$

where u_0' is the initial value for the particle velocity.

1.3.7.5. Small-amplitude waves in a strained medium

We now consider the propagation of small-amplitude elastic waves in a homogeneously strained medium. As defined previously, \bar{X} or a are the coordinates in the natural or unstressed state. X are the coordinates in the initial or homogeneously strained state. $u_i = x_i - X_i$ are the components of displacement from the initial state due to the wave.

Starting from (1.3.7.4), we get

$$T_{ij} = \frac{\rho_0}{J} \alpha_{ik} \alpha_{jl} \frac{\partial U}{\partial S_{kl}}.$$

Its partial derivative is

$$\frac{\partial T_{ij}}{\partial x_j} = \frac{1}{J} \frac{\partial}{\partial X_k} \left[\rho_0 \alpha_{il} \frac{\partial U}{\partial S_{kl}} \right].$$

If we expand the state function about the initial configuration, it follows that

$$\begin{aligned}\rho_0 U(X_k, S_{ij}) &= \rho_0 U(X_k) + c_{ij} S_{ij} + \frac{1}{2} c_{ijkl} S_{ij} S_{kl} \\ &\quad + \frac{1}{6} c_{ijklmn} S_{ij} S_{kl} S_{mn} + \dots\end{aligned}$$

The linearized stress derivatives become

$$\frac{\partial T_{ij}}{\partial x_j} = [c_{jl} \delta_{ik} + c_{ijkl}] \frac{\partial^2 x_k}{\partial X_j \partial X_l}.$$

If we let $D_{ijkl} = [c_{jl} \delta_{ik} + c_{ijkl}]$, the equation of motion in the initial state is

$$\rho_0 u_i'' = D_{jkli} \frac{\partial^2 u_k}{\partial X_j \partial X_l}.\tag{1.3.7.13}$$

The coefficients D_{ijkl} do not present the symmetry of the coefficients c_{ijkl} except in the natural state where D_{ijkl} and c_{ijkl} are equal.

The simplest solutions of the equation of motion are plane waves. We now assume plane sinusoidal waves of the form

$$u_i = A_i \exp[i(\omega t - \mathbf{k} \cdot \mathbf{X})],\tag{1.3.7.14}$$

where \mathbf{k} is the wavevector.

Substitution of (1.3.7.14) into (1.3.7.13) results in

$$\rho_0 \omega^2 A_j = D_{ijkl} k_j k_l A_k$$