t_{kl}, the thermodynamic tensor conjugate to the variable S_{kl}/ρ_0, is generally denoted as the 'second Piola–Kirchoff stress tensor'.

Using Φ, the strain energy per unit volume, Newton's law (1.3.7.4) takes the form

$$\rho x_i'' = \frac{\partial}{\partial X_j}\left(\alpha_{jk}\frac{\partial\Phi}{\partial S_{ik}}\right) \quad \text{or} \quad \rho u_i'' = \frac{\partial}{\partial X_j}\left(\alpha_{jk}\frac{\partial\Phi}{\partial S_{ik}}\right)$$

and

$$T_{ij} = \alpha_{jk}\frac{\partial\Phi}{\partial S_{ik}}. \qquad (1.3.7.5)$$

1.3.7.3. *Wave propagation in a nonlinear elastic medium*

As an example, let us consider the case of a plane finite amplitude wave propagating along the x_1 axis. The displacement components in this case become

$$u_1 = u_1(X_1, t); \quad u_2 = u_2(X_1, t); \quad u_3 = u_3(X_1, t).$$

Thus, the Jacobian matrix α_{ij} reduces to

$$J = \begin{pmatrix} \alpha_{11} & 0 & 0 \\ \alpha_{21} & 0 & 0 \\ \alpha_{31} & 0 & 0 \end{pmatrix}.$$

The Lagrangian strain matrix is [equation (1.3.6.8)]

$$S = \tfrac{1}{2}\left(J^TJ - \delta\right).$$

The only nonvanishing strain components are, therefore,

$$\begin{aligned} S_{11} &= \tfrac{1}{2}\left(\alpha_{11}^2 + \alpha_{21}^2 + \alpha_{31}^2\right) - 1 \\ &= \frac{\partial u_1}{\partial X_1} + \tfrac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right] \\ S_{12} &= S_{21} = \tfrac{1}{2}\frac{\partial u_2}{\partial X_1} \\ S_{13} &= S_{31} = \tfrac{1}{2}\frac{\partial u_3}{\partial X_1} \end{aligned}$$

and the strain invariants reduce to

$$I_1 = S_{11}; \quad I_2 = -(S_{12}S_{21} + S_{13}S_{31}); \quad I_3 = 0.$$

1.3.7.3.1. *Isotropic media*

In this case, the strain-energy density becomes

$$\begin{aligned} \Phi = {}& \tfrac{1}{2}(\lambda + 2\mu)(S_{11})^2 + 2\mu(S_{12}S_{21} + S_{13}S_{31}) + \tfrac{1}{3}(l + 2m)(S_{11})^3 \\ &+ 2mS_{11}(S_{12}S_{21} + S_{13}S_{31}). \end{aligned} \qquad (1.3.7.6)$$

Differentiating (1.3.7.6) with respect to the strains, we get

$$\begin{aligned} \frac{\partial\Phi}{\partial S_{11}} &= (\lambda + 2\mu)S_{11} + (l + 2m)(S_{11})^2 + 2m(S_{12}S_{21} + S_{13}S_{31}) \\ \frac{\partial\Phi}{\partial S_{12}} &= 2\mu S_{21} + 2mS_{11}S_{21} \\ \frac{\partial\Phi}{\partial S_{13}} &= 2\mu S_{31} + 2mS_{11}S_{31} \\ \frac{\partial\Phi}{\partial S_{21}} &= 2\mu S_{12} + 2mS_{11}S_{12} \\ \frac{\partial\Phi}{\partial S_{31}} &= 2\mu S_{13} + 2mS_{11}S_{13}. \end{aligned}$$

All the other $\partial\Phi/\partial S_{ij} = 0$.

From (1.3.7.5), we derive the stress components:

$$\begin{aligned} T_{11} &= \alpha_{1k}\frac{\partial\Phi}{\partial S_{1k}}; \quad T_{12} = \alpha_{2k}\frac{\partial\Phi}{\partial S_{1k}}; \quad T_{13} = \alpha_{3k}\frac{\partial\Phi}{\partial S_{1k}}; \\ T_{21} &= \alpha_{1k}\frac{\partial\Phi}{\partial S_{2k}}; \quad T_{22} = \alpha_{2k}\frac{\partial\Phi}{\partial S_{2k}}; \quad T_{23} = \alpha_{3k}\frac{\partial\Phi}{\partial S_{2k}}; \\ T_{31} &= \alpha_{1k}\frac{\partial\Phi}{\partial S_{3k}}; \quad T_{32} = \alpha_{2k}\frac{\partial\Phi}{\partial S_{3k}}; \quad T_{33} = \alpha_{3k}\frac{\partial\Phi}{\partial S_{3k}}. \end{aligned}$$

Note that this tensor is not symmetric.

For the particular problem discussed here, the three components of the equation of motion are

$$\begin{aligned} \rho u_1'' &= \mathrm{d}T_{11}/\mathrm{d}X_1, \\ \rho u_2'' &= \mathrm{d}T_{21}/\mathrm{d}X_1, \\ \rho u_3'' &= \mathrm{d}T_{31}/\mathrm{d}X_1. \end{aligned}$$

If we retain only terms up to the quadratic order in the displacement gradients, we obtain the following equations of motion:

$$\begin{aligned} \rho u_1'' = {}& (\lambda + 2\mu)\frac{\partial^2 u_1}{\partial X_1^2} + [3(\lambda + 2\mu) + 2(l + 2m)]\frac{\partial u_1}{\partial X_1}\frac{\partial^2 u_1}{\partial X_1^2} \\ &+ (\lambda + 2\mu + m)\left[\frac{\partial u_2}{\partial X_1}\frac{\partial^2 u_2}{\partial X_1^2} + \frac{\partial u_3}{\partial X_1}\frac{\partial^2 u_3}{\partial X_1^2}\right] \\ \rho u_2'' = {}& \mu\frac{\partial^2 u_2}{\partial X_1^2} + (\lambda + 2\mu + m)\left[\frac{\partial u_1}{\partial X_1}\frac{\partial^2 u_2}{\partial X_1^2} + \frac{\partial u_2}{\partial X_1}\frac{\partial^2 u_1}{\partial X_1^2}\right] \\ \rho u_3'' = {}& \mu\frac{\partial^2 u_3}{\partial X_1^2} + (\lambda + 2\mu + m)\left[\frac{\partial u_1}{\partial X_1}\frac{\partial^2 u_3}{\partial X_1^2} + \frac{\partial u_3}{\partial X_1}\frac{\partial^2 u_1}{\partial X_1^2}\right]. \end{aligned} \qquad (1.3.7.7)$$

1.3.7.3.2. *Cubic media (most symmetrical groups)*

In this case, the strain-energy density becomes

$$\begin{aligned} \Phi = {}& \tfrac{1}{2}c_{11}(S_{11})^2 + c_{44}\left[(S_{12})^2 + (S_{21})^2 + (S_{31})^2 + (S_{13})^2\right] \\ &+ c_{111}(S_{11})^3 + \tfrac{1}{2}c_{166}S_{11}\left[(S_{12})^2 + (S_{21})^2 + (S_{31})^2 + (S_{13})^2\right]. \end{aligned} \qquad (1.3.7.8)$$

Differentiating (1.3.7.8) with respect to the strain, one obtains

$$\begin{aligned} \frac{\partial\Phi}{S_{11}} &= c_{11}S_{11} + 3c_{111}(S_{11})^2 + \tfrac{1}{2}c_{166}[(S_{12})^2 + (S_{21})^2 \\ &\quad + (S_{31})^2 + (S_{13})^2)] \\ \frac{\partial\Phi}{S_{21}} &= 2c_{44}S_{21} + c_{166}S_{11}S_{21} \\ \frac{\partial\Phi}{S_{31}} &= 2c_{44}S_{31} + c_{166}S_{11}S_{31}. \end{aligned}$$

All other $\partial\Phi/S_{ij} = 0$. From (1.3.7.5), we derive the stress components:

$$\begin{aligned} T_{11} &= \alpha_{1k}\frac{\partial\Phi}{\partial S_{1k}} \\ T_{21} &= \alpha_{1k}\frac{\partial\Phi}{\partial S_{2k}} \\ T_{31} &= \alpha_{1k}\frac{\partial\Phi}{\partial S_{3k}}. \end{aligned}$$

In this particular case, the three components of the equation of motion are