

1. TENSORIAL ASPECTS OF PHYSICAL PROPERTIES

$$\begin{aligned} \rho u_1'' &= dT_{11}/dX_1 \\ \rho u_2'' &= dT_{21}/dX_1 \\ \rho u_3'' &= dT_{31}/dX_1. \end{aligned}$$

If we retain only terms up to the quadratic order in the displacement gradients, we obtain the following equations of motion:

$$\begin{aligned} \rho u_1'' &= c_{11} \frac{\partial^2 u_1}{\partial X_1^2} + [3c_{11} + c_{111}] \frac{\partial u_1}{\partial X_1} \frac{\partial^2 u_1}{\partial X_1^2} \\ &\quad + (c_{11} + c_{166}) \left[\frac{\partial u_2}{\partial X_1} \frac{\partial^2 u_2}{\partial X_1^2} + \frac{\partial u_3}{\partial X_1} \frac{\partial^2 u_3}{\partial X_1^2} \right] \\ \rho u_2'' &= c_{44} \frac{\partial^2 u_2}{\partial X_1^2} + (c_{11} + c_{166}) \left[\frac{\partial u_1}{\partial X_1} \frac{\partial^2 u_2}{\partial X_1^2} + \frac{\partial u_2}{\partial X_1} \frac{\partial^2 u_1}{\partial X_1^2} \right] \\ \rho u_3'' &= c_{44} \frac{\partial^2 u_3}{\partial X_1^2} + (c_{11} + c_{166}) \left[\frac{\partial u_1}{\partial X_1} \frac{\partial^2 u_3}{\partial X_1^2} + \frac{\partial u_3}{\partial X_1} \frac{\partial^2 u_1}{\partial X_1^2} \right], \end{aligned} \tag{1.3.7.9}$$

which are identical to (1.3.7.7) if we put

$$c_{11} = \lambda + 2\mu; \quad c_{44} = \mu; \quad c_{111} = 2(l + 2m); \quad c_{166} = m.$$

1.3.7.4. Harmonic generation

The coordinates in the medium free of stress are denoted either a or \bar{X} . The notation \bar{X} is used when we have to discriminate the natural configuration, \bar{X} , from the initial configuration X . Here, the process that we describe refers to the propagation of an elastic wave in a medium free of stress (natural state) and the coordinates will be denoted a_i .

Let us first examine the case of a pure longitudinal mode, *i.e.*

$$u_1 = u_1(a_1, t); \quad u_2 = u_3 = 0.$$

The equations of motion, (1.3.7.7) and (1.3.7.9), reduce to

$$\rho u_1'' = (\lambda + 2\mu) \frac{\partial^2 u_1}{\partial a_1^2} + [3(\lambda + 2\mu) + 2(l + 2m)] \frac{\partial u_1}{\partial a_1} \frac{\partial^2 u_1}{\partial a_1^2}$$

for an isotropic medium or

$$\rho u_1'' = c_{11} \frac{\partial^2 u_1}{\partial a_1^2} + [3c_{11} + c_{166}] \frac{\partial u_1}{\partial a_1} \frac{\partial^2 u_1}{\partial a_1^2}$$

for a cubic crystal (most symmetrical groups) when a pure longitudinal mode is propagated along [100].

For both cases, we have a one-dimensional problem; (1.3.7.7) and (1.3.7.9) can therefore be written

$$\rho u_1'' = K_2 \frac{\partial^2 u_1}{\partial a_1^2} + [3K_2 + K_3] \frac{\partial u_1}{\partial a_1} \frac{\partial^2 u_1}{\partial a_1^2}. \tag{1.3.7.10}$$

The same equation is also valid when a pure longitudinal mode is propagated along [110] and [111], with the following correspondence:

$$\begin{aligned} [100] \quad K_2 &= c_{11}, \quad K_3 = c_{111} \\ [110] \quad K_2 &= \frac{c_{11} + c_{12} + 2c_{44}}{2}, \quad K_3 = \frac{c_{111} + 3c_{112} + 12c_{166}}{4} \\ [111] \quad K_2 &= \frac{c_{11} + 2c_{12} + 4c_{44}}{3}, \\ K_3 &= \frac{c_{111} + 6c_{112} + 12c_{144} + 24c_{166} + 2c_{123} + 16c_{456}}{9}. \end{aligned}$$

Let us assume that $K_3 \ll K_2$; a perturbation solution to (1.3.7.10) is

$$u = u^0 + u^1,$$

where $u^1 \ll u^0$ with

$$u^0 = A \sin(ka - \omega t) \tag{1.3.7.11}$$

$$u^1 = Ba \sin 2(ka - \omega t) + Ca \cos 2(ka - \omega t). \tag{1.3.7.12}$$

If we substitute the trial solutions into (1.3.7.10), we find after one iteration the following approximate solution:

$$u = A \sin(ka - \omega t) - \frac{(kA)^2(3K_2 + K_3)}{8\rho c^2} a \cos 2(ka - \omega t),$$

which involves second-harmonic generation.

If additional iterations are performed, higher harmonic terms will be obtained. A well known property of the first-order nonlinear equation (1.3.7.10) is that its solutions exhibit discontinuous behaviour at some point in space and time. It can be seen that such a discontinuity would appear at a distance from the origin given by (Breazeale, 1984)

$$L = -2 \frac{(K_2)^2}{3K_2 + K_3} \rho \omega u_0',$$

where u_0' is the initial value for the particle velocity.

1.3.7.5. Small-amplitude waves in a strained medium

We now consider the propagation of small-amplitude elastic waves in a homogeneously strained medium. As defined previously, \bar{X} or a are the coordinates in the natural or unstressed state. X are the coordinates in the initial or homogeneously strained state. $u_i = x_i - X_i$ are the components of displacement from the initial state due to the wave.

Starting from (1.3.7.4), we get

$$T_{ij} = \frac{\rho_0}{J} \alpha_{ik} \alpha_{jl} \frac{\partial U}{\partial S_{kl}}.$$

Its partial derivative is

$$\frac{\partial T_{ij}}{\partial x_j} = \frac{1}{J} \frac{\partial}{\partial X_k} \left[\rho_0 \alpha_{il} \frac{\partial U}{\partial S_{kl}} \right].$$

If we expand the state function about the initial configuration, it follows that

$$\begin{aligned} \rho_0 U(X_k, S_{ij}) &= \rho_0 U(X_k) + c_{ij} S_{ij} + \frac{1}{2} c_{ijkl} S_{ij} S_{kl} \\ &\quad + \frac{1}{6} c_{ijklmn} S_{ij} S_{kl} S_{mn} + \dots \end{aligned}$$

The linearized stress derivatives become

$$\frac{\partial T_{ij}}{\partial x_j} = [c_{jl} \delta_{ik} + c_{ijkl}] \frac{\partial^2 x_k}{\partial X_j \partial X_l}.$$

If we let $D_{ijkl} = [c_{jl} \delta_{ik} + c_{ijkl}]$, the equation of motion in the initial state is

$$\rho_0 u_i'' = D_{jkli} \frac{\partial^2 u_k}{\partial X_j \partial X_l}. \tag{1.3.7.13}$$

The coefficients D_{ijkl} do not present the symmetry of the coefficients c_{ijkl} except in the natural state where D_{ijkl} and c_{ijkl} are equal.

The simplest solutions of the equation of motion are plane waves. We now assume plane sinusoidal waves of the form

$$u_i = A_i \exp[i(\omega t - \mathbf{k} \cdot \mathbf{X})], \tag{1.3.7.14}$$

where \mathbf{k} is the wavevector.

Substitution of (1.3.7.14) into (1.3.7.13) results in

$$\rho_0 \omega^2 A_j = D_{ijkl} k_j k_l A_k$$