

1.3. ELASTIC PROPERTIES

replaces in (1.3.3.8) the stress distribution by a hydrostatic pressure, one obtains for the components of the strain tensor

$$\begin{aligned} S_1 &= -p(s_{11} + s_{12} + s_{13}) \\ S_2 &= -p(s_{12} + s_{22} + s_{23}) \\ S_3 &= -p(s_{13} + s_{23} + s_{33}). \end{aligned}$$

From this, we deduce the volume compressibility,  $\chi$ , which is the inverse of the *bulk modulus*,  $\kappa$ :

$$\chi = \kappa^{-1} = -\frac{1}{p} \frac{\delta V}{V} = s_{11} + s_{22} + s_{33} + 2(s_{12} + s_{23} + s_{13}). \tag{1.3.3.11}$$

This expression reduces for a cubic or isotropic medium to

$$\chi = \kappa^{-1} = 3(s_{11} + 2s_{12}). \tag{1.3.3.12}$$

1.3.3.4.2. Linear compressibility

Under the action of a hydrostatic pressure, each vector assumes a different elongation. This elongation is given by equation (1.3.1.6):

$$\frac{u_r}{r} = \frac{S_{ij}x_i x_j}{r^2} = S_{ij}\alpha_i \alpha_j = s_{ijkl} T_{kl} \alpha_i \alpha_j,$$

where the  $\alpha_i$ 's are the direction cosines of  $\mathbf{r}$ . The coefficient of linear compressibility is, by definition,  $(-1/p)(u_r/r)$ . Replacing  $T_{kl}$  by its value  $-p\delta_{kl}$ , we obtain for the coefficient of linear compressibility

$$-\frac{1}{p} \frac{u_r}{r} = s_{ijkl} \alpha_i \alpha_j.$$

In the case of a cubic or isotropic medium, this expression reduces to

$$-\frac{1}{p} \frac{u_r}{r} = s_{11} + 2s_{12}.$$

The coefficient of linear compressibility is then equal to one third of the coefficient of volume compressibility. We note that the quadric of elongations is a sphere.

1.3.3.4.3. Young's modulus, Poisson's ratio

If the applied stress reduces to a uniaxial stress,  $T_{11} = T$ , the strain tensor is of the form

$$S_\alpha = s_{1\alpha} T.$$

In particular,

$$S_1 = s_{11} T; \quad S_2 = s_{12} T.$$

We deduce from this that Young's modulus (equation 1.3.3.1) is

$$E = 1/s_{11}. \tag{1.3.3.13}$$

The elongation of a bar under the action of a uniaxial stress is characterized by  $S_1$  and the diminution of the cross section is characterized by  $S_2$  and  $S_3$ . For a cubic material, the relative diminution of the diameter is

$$S_2 = S_3 = s_{12} T.$$

One deduces from this that  $s_{12}$  is necessarily of opposite sign to  $s_{11}$  and one calls the ratio  $\nu = -s_{12}/s_{11}$  *Poisson's ratio*.

Putting this value into expression (1.3.3.12) for the coefficient of compressibility in cubic or isotropic materials gives

$$\kappa^{-1} = 3s_{11}(1 - 2\nu). \tag{1.3.3.14}$$

As the coefficient of compressibility, by definition, is always positive, we have

$$0 < \nu < 0.5.$$

In practice, Poisson's ratio is always close to 0.3. It is a dimensionless number. The quantity  $s_{44}/2(s_{11} - s_{12})$  represents the departure from isotropy of the material and is the anisotropy factor. It is to be noted that cubic materials are not isotropic for elastic properties. Table 1.3.3.2 gives the values of  $s_{11}$ ,  $s_{12}$ ,  $s_{44}$ ,  $\nu$  and  $s_{44}/2(s_{11} - s_{12})$  for a few cubic materials.

1.3.3.4.4. Variation of Young's modulus with orientation

It is interesting to calculate Young's modulus in any direction. For this it is sufficient to change the axes of the tensor  $s_{ijkl}$ . If  $A$  is the matrix associated with the change of axes, leading to the direction  $x_1$  changing to the direction  $x'_1$ , then Young's modulus in this new direction is

$$E' = 1/s'_{11}$$

with

$$s'_{11} = s'_{1111} = A_{1i} A_{1j} A_{1k} A_{1l} s_{ijkl}. \tag{1.3.3.15}$$

Table 1.3.3.2. Elastic compliances of some cubic materials in (GPa)<sup>-1</sup> (after Landolt-Börnstein, 1979)

Material	$s_{11}$	$s_{12}$	$s_{44}$	$\nu$	$s_{44}/2(s_{11} - s_{12})$
Ag	22.9	-9.8	22.1	0.428	0.338
Al	16.0	-5.8	35.3	0.362	0.810
C (diamond)	1.12	-0.14	1.83	0.125	0.726
Cu	15.0	-6.3	13.3	0.42	0.312
Fe	7.67	-2.83	8.57	0.369	0.408
Ge	9.73	-2.64	14.9	0.271	0.602
Mo	2.71	-0.74	9.00	0.273	1.304
Ni	7.67	-2.93	8.23	0.382	0.388
Pb	93.7	-43.04	68.0	0.459	0.249
Si	7.74	-2.16	12.60	0.279	0.636
W	2.49	-0.70	6.35	0.281	0.995
LiF	11.6	-3.35	15.8	0.289	0.528
MgO	4.01	-0.96	6.47	0.239	0.651
NaCl	23.9	-3.20	78.7	0.133	1.452
GaAs	11.75	-3.66	16.8	0.311	0.545
ZnS	19.7	-7.6	22.6	0.386	0.414
BaTiO <sub>3</sub>	8.33	-2.68	9.24	0.322	0.420
Adamantane	240	-79	295	0.329	0.462
Almandine (Fe <sub>3</sub> Al <sub>2</sub> Si <sub>3</sub> O <sub>12</sub> )	4.036	-1.093	10.77	0.271	1.050
Spinel (MgAl <sub>2</sub> O <sub>4</sub> )	5.80	-2.05	6.49	0.353	0.413