International Tables for Crystallography (2006). Vol. D, Section 1.5.2, pp. 109–116.

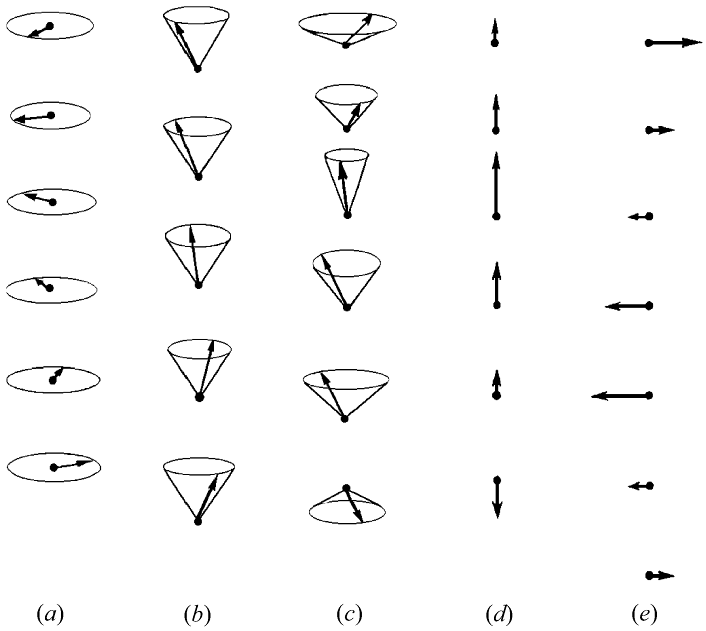

Fig. 1.5.1.4. Helical and sinusoidal magnetic structures. (*a*) An antiferromagnetic helix; (*b*) a cone spiral; (*c*) a cycloidal spiral; (*d*) a longitudinal spin-density wave; (*e*) a transverse spin-density wave.

the vectors of the magnetization of the layers are arranged on the surface of a cone. The ferromagnetic magnetization is aligned along the z axis. This structure is called a ferromagnetic helix. It usually belongs to the incommensurate magnetic structures.

More complicated antiferromagnetic structures also exist: sinusoidal structures, which also consist of layers in which all the magnetic moments are parallel to each other. Fig. 1.5.1.4(*c*) displays the cycloidal spiral and Figs. 1.5.1.4(*d*) and (*e*) display longitudinal and transverse spin density waves, respectively.

1.5.2. Magnetic symmetry

As discussed in Section 1.5.1, in studies of the symmetry of magnetics one should take into account not only the crystallographic elements of symmetry (rotations, reflections and translations) but also the time-inversion element, which causes the reversal of the magnetic moment density vector $\mathbf{m}(\mathbf{r})$. Following Landau & Lifshitz (1957), we shall denote this element by R. If combined with any crystallographic symmetry element G we get a product RG, which some authors call the space-time symmetry operator. We shall not use this terminology in the following.

To describe the symmetry properties of magnetics, one should use magnetic point and space groups instead of crystallographic ones. (See also Section 1.2.5.)

By investigating the 'four-dimensional groups of three-dimensional space', Heesch (1930) found not only the 122 groups that now are known as magnetic point groups but also the seven triclinic and 91 monoclinic magnetic space groups. He also recognized that these groups can be used to describe the symmetry of spin arrangements. The present interest in magnetic symmetry was much stimulated by Shubnikov (1951), who considered the symmetry groups of figures with black and white faces, which he called antisymmetry groups. The change of colour of the faces in antisymmetry (black–white symmetry, see also Section 3.3.5) corresponds to the element R. These antisymmetry classes were derived as magnetic symmetry point groups by Tavger & Zaitsev (1956). Beside antisymmetry, the concept of colour (or generalized) symmetry also was developed, in which the number of colours is not 2 but 3, 4 or 6 (see Belov *et al.*, 1964; Koptsik & Kuzhukeev, 1972). A different generalization to more than two colours was proposed by van der Waerden & Burckhardt (1961). The various approaches have been compared by Schwarzenberger (1984).

Table 1.5.2.1. *Comparison of different symbols for magnetic point groups*

Schoenflies	Hermann–Mauguin	Shubnikov	
$\boldsymbol{D}_{4R}$	$4221'$	$4{:}21'$	$4{:}2\underline{1}$
$\boldsymbol{D}_4$	422	$4{:}2$	$4{:}2$
$\boldsymbol{D}_4(\boldsymbol{C}_4)$	$42'2'$	$4{:}2'$	$4{:}\underline{2}$
$\boldsymbol{D}_4(\boldsymbol{D}_2)$	$4'22'$	$4'{:}2$	$\underline{4}{:}2$

As the theories of antisymmetry and of magnetic symmetry evolved often independently, different authors denote the operation of time inversion (black–white exchange) by different symbols. Of the four frequently used symbols ($R = E' = \underline{1} = 1'$) we shall use in this article only two: R and $1'$.

1.5.2.1. *Magnetic point groups*

Magnetic point groups may contain rotations, reflections, the element R and their combinations. A set of such elements that satisfies the group properties is called a magnetic point group. It is obvious that there are 32 trivial magnetic point groups; these are the ordinary crystallographic point groups supplemented by the element R. Each of these point groups contains all the elements of the ordinary point group $\mathcal{P}$ and also all the elements of this group $\mathcal{P}$ multiplied by R. This type of magnetic point group $\mathcal{M}_{P1}$ can be represented by

$$\mathcal{M}_{P1} = \mathcal{P} + R\mathcal{P}. \qquad (1.5.2.1)$$

These groups are sometimes called 'grey' magnetic point groups. As pointed out above, all dia- and paramagnets belong to this type of point group. To this type belong also antiferromagnets with a magnetic space group that contains translations multiplied by R (space groups of type III^b).

The second type of magnetic point group, which is also trivial in some sense, contains all the 32 crystallographic point groups without the element R in any form. For this type $\mathcal{M}_{P2} = \mathcal{P}$. Thirteen of these point groups allow ferromagnetic spontaneous magnetization (ferromagnetism, ferrimagnetism, weak ferromagnetism). They are listed in Table 1.5.2.4. The remaining 19 point groups describe antiferromagnets. The groups $\mathcal{M}_{P2}$ are often called 'white' magnetic point groups.

The third type of magnetic point group $\mathcal{M}_{P3}$, 'black and white' groups (which are the only nontrivial ones), contains those point groups in which R enters only in combination with rotations or reflections. There are 58 point groups of this type. Eighteen of them describe different types of ferromagnetism (see Table 1.5.2.4) and the others represent antiferromagnets.

Replacing R by the identity element E in the magnetic point groups of the third type does not change the number of elements in the point group. Thus each group of the third type $\mathcal{M}_{P3}$ is isomorphic to a group $\mathcal{P}$ of the second type.

The method of derivation of the nontrivial magnetic groups given below was proposed by Indenbom (1959). Let $\mathcal{H}$ denote the set of those elements of the group $\mathcal{P}$ which enter into the asso-

Table 1.5.2.2. *Comparison of different symbols for the elements of magnetic point groups*

Magnetic point group	Elements: Schoenflies	Elements: Hermann–Mauguin
$\boldsymbol{D}_{4R} = 4221'$	$E, C_2, 2C_4, 2U_2, 2U_2^a,$ $R, RC_2, 2RC_4, 2RU_2, 2RU_2^a$	$1, 2_x, 2_y, 2_z, 2_{xy}, 2_{-xy}, \pm 4_z,$ $1', 2'_x, 2'_y, 2'_z, 2'_{xy}, 2'_{-xy}, \pm 4'_z$
$\boldsymbol{D}_4 = 422$	$E, C_2, 2C_4, 2U_2, 2U_2^a$	$1, 2_x, 2_y, 2_z, 2_{xy}, 2_{-xy}, \pm 4_z$
$\boldsymbol{D}_4(\boldsymbol{C}_4) = 42'2'$	$E, C_2, 2C_4, 2RU_2, 2RU_2^a$	$1, 2_z, \pm 4_z, 2'_x, 2'_y, 2'_{xy}, 2'_{-xy}$
$\boldsymbol{D}_4(\boldsymbol{D}_2) = 4'22'$	$E, C_2^z, C_2^y, C_2^x, 2RU_2^a, 2RC_4$	$1, 2_x, 2_y, 2_z, 2'_{xy}, 2'_{-xy}, \pm 4'_z$

Table 1.5.2.3. *The 90 magnetic point groups of types 2 and 3*

1	2	3	4	5	6
	Symbol of magnetic point group $\mathcal{M}$				
			Hermann–Mauguin		
System	Schoenflies	Shubnikov	Short	Full	Symmetry operators of group $\mathcal{M}$
Triclinic	$\boldsymbol{C}_1$	1	1	1	1
	$\boldsymbol{C}_i$	$\bar{2}$	$\bar{1}$	$\bar{1}$	$1, \bar{1}$
	$\boldsymbol{C}_i(\boldsymbol{C}_1)$	$\underline{\bar{2}}$	$\bar{1}'$	$\bar{1}'$	$1, \bar{1}'$
Monoclinic	$\boldsymbol{C}_2$	2	2	121	$1, 2_y$
	$\boldsymbol{C}_2(\boldsymbol{C}_1)$	$\underline{2}$	$2'$	$12'1$	$1, 2'_y$
	$\boldsymbol{C}_s$	m	m	$1m1$	$1, m_y$
	$\boldsymbol{C}_s(\boldsymbol{C}_1)$	$\underline{m}$	m'	$1m'1$	$1, m'_y$
	$\boldsymbol{C}_{2h}$	$2 : m$	$2/m$	$1\frac{2}{m}1$	$1, \bar{1}, 2_y, m_y$
	$\boldsymbol{C}_{2h}(\boldsymbol{C}_i)$	$\underline{2} : \underline{m}$	$2'/m'$	$1\frac{2'}{m'}1$	$1, \bar{1}, 2'_y, m'_y$
	$\boldsymbol{C}_{2h}(\boldsymbol{C}_2)$	$2 : \underline{m}$	$2/m'$	$1\frac{2}{m'}1$	$1, 2_y, \bar{1}', m'_y$
	$\boldsymbol{C}_{2h}(\boldsymbol{C}_s)$	$\underline{2} : m$	$2'/m$	$1\frac{2'}{m}1$	$1, m_y, \bar{1}', 2'_y$
Orthorhombic	$\boldsymbol{D}_2$	$2 : 2$	222	222	$1, 2_x, 2_y, 2_z$
	$\boldsymbol{D}_2(\boldsymbol{C}_2)$	$2 : \underline{2}$	$2'2'2$	$2'2'2$	$1, 2_z, 2'_x, 2'_y$
	$\boldsymbol{C}_{2v}$	$2{\cdot}m$	$mm2$	$mm2$	$1, 2_z, m_x, m_y$
	$\boldsymbol{C}_{2v}(\boldsymbol{C}_2)$	$2{\cdot}\underline{m}$	$m'm'2$	$m'm'2$	$1, 2_z, m'_x, m'_y$
	$\boldsymbol{C}_{2v}(\boldsymbol{C}_s)$	$\underline{2}{\cdot}m$	$2'm'm$	$2'm'm$	$1, m_z, 2'_x, m'_y$
	$\boldsymbol{D}_{2h}$	$m{\cdot}2 : m$	mmm	$\frac{2}{m}\frac{2}{m}\frac{2}{m}$	$1, \bar{1}, 2_x, 2_y, 2_z, m_x, m_y, m_z$
	$\boldsymbol{D}_{2h}(\boldsymbol{C}_{2h})$	$\underline{m}{\cdot}2 : m$	$mm'm'$	$\frac{2}{m}\frac{2'}{m'}\frac{2'}{m'}$	$1, \bar{1}, 2_x, m_x, 2'_y, 2'_z, m'_y, m'_z$
	$\boldsymbol{D}_{2h}(\boldsymbol{D}_2)$	$\underline{m}{\cdot}2 : \underline{m}$	$m'm'm'$	$\frac{2}{m'}\frac{2}{m'}\frac{2}{m'}$	$1, 2_x, 2_y, 2_z, \bar{1}', m'_x, m'_y, m'_z$
	$\boldsymbol{D}_{2h}(\boldsymbol{C}_{2v})$	$m{\cdot}2 : \underline{m}$	mmm'	$\frac{2'}{m}\frac{2'}{m}\frac{2}{m'}$	$1, 2_z, m_x, m_y, \bar{1}', 2'_x, 2'_y, m'_z$
Tetragonal	$\boldsymbol{C}_4$	4	4	4	$1, 2_z, \pm 4_z$
	$\boldsymbol{C}_4(\boldsymbol{C}_2)$	$\underline{4}$	$4'$	$4'$	$1, 2_z, \pm 4'_z$
	$\boldsymbol{S}_4$	$\bar{4}$	$\bar{4}$	$\bar{4}$	$1, 2_z, \pm\bar{4}_z$
	$\boldsymbol{S}_4(\boldsymbol{C}_2)$	$\underline{\bar{4}}$	$\bar{4}'$	$\bar{4}'$	$1, 2_z, \pm\bar{4}'_z$
	$\boldsymbol{C}_{4h}$	$4 : m$	$4/m$	$\frac{4}{m}$	$1, \bar{1}, 2_z, m_z, \pm 4_z, \pm\bar{4}_z$
	$\boldsymbol{C}_{4h}(\boldsymbol{C}_{2h})$	$\underline{4} : m$	$4'/m$	$\frac{4'}{m}$	$1, \bar{1}, 2_z, m_z, \pm 4'_z, \pm\bar{4}'_z$
	$\boldsymbol{C}_{4h}(\boldsymbol{C}_4)$	$4 : \underline{m}$	$4/m'$	$\frac{4}{m'}$	$1, 2_z, \pm 4_z, \bar{1}', m'_z, \pm\bar{4}'_z$
	$\boldsymbol{C}_{4h}(\boldsymbol{S}_4)$	$\underline{4} : \underline{m}$	$4'/m'$	$\frac{4'}{m'}$	$1, 2_z, \pm\bar{4}_z, \bar{1}', m'_z, \pm 4'_z$
	$\boldsymbol{D}_4$	$4 : 2$	422	422	$1, 2_x, 2_y, 2_z, 2_{xy}, 2_{-xy}, \pm 4_z$
	$\boldsymbol{D}_4(\boldsymbol{D}_2)$	$\underline{4} : 2$	$4'22'$	$4'22'$	$1, 2_x, 2_y, 2_z, 2'_{xy}, 2'_{-xy}, \pm 4'_z$
	$\boldsymbol{D}_4(\boldsymbol{C}_4)$	$4 : \underline{2}$	$42'2'$	$42'2'$	$1, 2_z, \pm 4_z, 2'_x, 2'_y, 2'_{xy}, 2'_{-xy}$
	$\boldsymbol{C}_{4v}$	$4{\cdot}m$	$4mm$	$4mm$	$1, 2_z, m_x, m_y, m_{xy}, m_{-xy}, \pm 4_z$
	$\boldsymbol{C}_{4v}(\boldsymbol{C}_{2v})$	$\underline{4}{\cdot}m$	$4'mm'$	$4'mm'$	$1, 2_z, m_x, m_y, m'_{xy}, m'_{-xy}, \pm 4'_z$
	$\boldsymbol{C}_{4v}(\boldsymbol{C}_4)$	$4{\cdot}\underline{m}$	$4m'm'$	$4m'm'$	$1, 2_z, \pm 4_z, m'_x, m'_y, m'_{xy}, m'_{-xy}$
	$\boldsymbol{D}_{2d}$	$\bar{4}{\cdot}m$	$\bar{4}2m$	$\bar{4}2m$	$1, 2_x, 2_y, 2_z, m_{xy}, m_{-xy}, \pm\bar{4}_z$
	$\boldsymbol{D}_{2d}(\boldsymbol{D}_2)$	$\underline{\bar{4}}{\cdot}\underline{m}$	$\bar{4}'2m'$	$\bar{4}'2m'$	$1, 2_x, 2_y, 2_z, m'_{xy}, m'_{-xy}, \pm\bar{4}'_z$
	$\boldsymbol{D}_{2d}(\boldsymbol{C}_{2v})$	$\underline{\bar{4}}{\cdot}m$	$\bar{4}'m2'$	$\bar{4}'m2'$	$1, 2_z, m_x, m_y, 2'_{xy}, 2'_{-xy}, \pm\bar{4}'_z$
	$\boldsymbol{D}_{2d}(\boldsymbol{S}_4)$	$\bar{4}{\cdot}\underline{m}$	$\bar{4}2'm'$	$\bar{4}2'm'$	$1, 2_z, \pm\bar{4}_z, 2'_x, 2'_y, m'_{xy}, m'_{-xy}$
	$\boldsymbol{D}_{4h}$	$m{\cdot}4 : m$	$4/mmm$	$\frac{4}{m}\frac{2}{m}\frac{2}{m}$	$1, \bar{1}, 2_x, 2_y, 2_z, 2_{xy}, 2_{-xy}, m_x, m_y, m_z, m_{xy}, m_{-xy}, \pm 4_z, \pm\bar{4}_z$
	$\boldsymbol{D}_{4h}(\boldsymbol{D}_{2h})$	$m{\cdot}\underline{4} : m$	$4'/mmm'$	$\frac{4'}{m}\frac{2}{m}\frac{2'}{m'}$	$1, \bar{1}, 2_x, 2_y, 2_z, m_x, m_y, m_z, 2'_{xy}, 2'_{-xy}, m'_{xy}, m'_{-xy}, \pm 4'_z, \pm\bar{4}'_z$
	$\boldsymbol{D}_{4h}(\boldsymbol{C}_{4h})$	$\underline{m}{\cdot}4 : m$	$4/mm'm'$	$\frac{4}{m}\frac{2'}{m'}\frac{2'}{m'}$	$1, \bar{1}, 2_z, m_z, \pm 4_z, \pm\bar{4}_z, 2'_x, 2'_y, 2'_{xy}, 2'_{-xy}, m'_x, m'_y, m'_{xy}, m'_{-xy}$
	$\boldsymbol{D}_{4h}(\boldsymbol{D}_4)$	$\underline{m}{\cdot}4 : \underline{m}$	$4/m'm'm'$	$\frac{4}{m'}\frac{2}{m'}\frac{2}{m'}$	$1, 2_x, 2_y, 2_z, 2_{xy}, 2_{-xy}, \pm 4_z, \bar{1}', m'_x, m'_y, m'_z, m'_{xy}, m'_{-xy}, \pm\bar{4}'_z$
	$\boldsymbol{D}_{4h}(\boldsymbol{C}_{4v})$	$m{\cdot}4 : \underline{m}$	$4/m'mm$	$\frac{4}{m'}\frac{2'}{m}\frac{2'}{m}$	$1, 2_z, m_x, m_y, m_{xy}, m_{-xy}, \pm 4_z, \bar{1}', 2'_x, 2'_y, 2'_{xy}, 2'_{-xy}, m'_z, \pm\bar{4}'_z$
	$\boldsymbol{D}_{4h}(\boldsymbol{D}_{2d})$	$m{\cdot}\underline{4} : \underline{m}$	$4'/m'm'm$	$\frac{4'}{m'}\frac{2}{m'}\frac{2'}{m}$	$1, 2_x, 2_y, 2_z, m_{xy}, m_{-xy}, \pm\bar{4}_z, \bar{1}', 2'_{xy}, 2'_{-xy}, m'_x, m'_y, m'_z, \pm 4'_z$

ciated magnetic group $\mathcal{M}_{P3}$ not multiplied by R. The set $\mathcal{H}$ contains the identity element E, for each element H also its inverse H^{-1}, and for each pair H_1, H_2 also its products H_1H_2 and H_2H_1. Thus the set $\mathcal{H}$ forms a group. It is a subgroup of the crystallographic group $\mathcal{P}$. Let P_i denote an element of $(\mathcal{P} - \mathcal{H})$. All these elements enter $\mathcal{M}_{P3}$ in the form of products RP_i because $RP_i = P_iR$ and $R^2 = 1$. Multiplying the elements of $\mathcal{M}_{P3}$ by a fixed element RP_1 corresponds to a permutation of the elements of $\mathcal{M}_{P3}$. This permutation maps each element of the subgroup $\mathcal{H}$ on an element of $\mathcal{M}_{P3}$ that does not belong to $\mathcal{H}$ and *vice versa*. It follows that one half of the elements of $\mathcal{M}_{P3}$ are elements of $(\mathcal{P} - \mathcal{H})$ multiplied by R and the other half belong to $\mathcal{H}$. The relation for the magnetic point groups of the third type may therefore be written as

$$\mathcal{M}_{P3} = \mathcal{H} + R(\mathcal{P} - \mathcal{H}) = \mathcal{H} + RP_1\mathcal{H}. \qquad (1.5.2.2)$$

$\mathcal{H}$ is therefore a subgroup of index 2 of $\mathcal{P}$. The subgroups of index 2 of $\mathcal{P}$ can easily be found using the tables of irreducible representations of the point groups. Every real non-unit one-dimensional representation of $\mathcal{P}$ contains equal numbers of characters $+1$ and -1. In the corresponding magnetic point group $\mathcal{M}_{P3}$, the elements of $\mathcal{P}$ with character -1 are multiplied by R and those with character $+1$ remain unchanged. The latter form the subgroup $\mathcal{H}$. This rule can be stated as a theorem: every real non-unit one-dimensional representation τ of a point group of symmetry $\mathcal{P}$ produces an isomorphic mapping of this group upon a magnetic group $\mathcal{M}_{P3}$ (Indenbom, 1959). This concept will be developed in Section 1.5.3.

Using the Schoenflies symbols and the method described above, the point groups of magnetic symmetry (magnetic point groups) can be denoted by $\mathcal{P}(\mathcal{H})$, where $\mathcal{P}$ is the symbol of the original crystallographic point group and $\mathcal{H}$ is the symbol of that

Table 1.5.2.3 (*cont.*)

1	2	3	4	5	6
	Symbol of magnetic point group $\mathcal{M}$				
			Hermann–Mauguin		
System	Schoenflies	Shubnikov	Short	Full	Symmetry operators of group $\mathcal{M}$
Trigonal	$\boldsymbol{C}_3$	3	3	3	$1, \pm 3_z$
	$\boldsymbol{S}_6$	$\bar{6}$	$\bar{3}$	$\bar{3}$	$1, \bar{1}, \pm 3_z, \pm\bar{3}_z$
	$\boldsymbol{S}_6(\boldsymbol{C}_3)$	$\underline{\bar{6}}$	$\bar{3}'$	$\bar{3}'$	$1, \pm 3_z, \bar{1}', \pm\bar{3}'_z$
	$\boldsymbol{D}_3$	$3:2$	32	321	$1, 3(2_\perp), \pm 3_z$
	$\boldsymbol{D}_3(\boldsymbol{C}_3)$	$3:\underline{2}$	$32'$	$32'1$	$1, \pm 3_z, 3(2'_\perp)$
	$\boldsymbol{C}_{3v}$	$3\cdot m$	$3m$	$3m1$	$1, 3(m_\perp), \pm 3_z$
	$\boldsymbol{C}_{3v}(\boldsymbol{C}_3)$	$3\cdot\underline{m}$	$3m'$	$3m'1$	$1, \pm 3_z, 3(m'_\perp)$
	$\boldsymbol{D}_{3d}$	$\bar{6}\cdot m$	$\bar{3}m$	$\bar{3}\frac{2}{m}1$	$1, \bar{1}, 3(2_\perp), 3(m_\perp), \pm 3_z, \pm\bar{3}_z$
	$\boldsymbol{D}_{3d}(\boldsymbol{S}_6)$	$\bar{6}\cdot\underline{m}$	$\bar{3}m'$	$\bar{3}\frac{2'}{m'}1$	$1, \bar{1}, \pm 3_z, \pm\bar{3}_z, 3(2'_\perp), 3(m'_\perp)$
	$\boldsymbol{D}_{3d}(\boldsymbol{D}_3)$	$\underline{\bar{6}}\cdot\underline{m}$	$\bar{3}'m'$	$\bar{3}'\frac{2}{m'}1$	$1, 3(2_\perp), \pm 3_z, \bar{1}', 3(m'_\perp), \pm\bar{3}'_z$
	$\boldsymbol{D}_{3d}(\boldsymbol{C}_{3v})$	$\underline{\bar{6}}\cdot m$	$\bar{3}'m$	$\bar{3}'\frac{2'}{m}1$	$1, 3(m_\perp), \pm 3_z, \bar{1}', 3(2'_\perp), \pm\bar{3}'_z$
Hexagonal	$\boldsymbol{C}_6$	6	6	6	$1, 2_z, \pm 3_z, \pm 6_z$
	$\boldsymbol{C}_6(\boldsymbol{C}_3)$	$\underline{6}$	$6'$	$6'$	$1, \pm 3_z, 2'_z, \pm 6'_z$
	$\boldsymbol{C}_{3h}$	$3:m$	$\bar{6}$	$\bar{6}$	$1, m_z, \pm 3_z, \pm\bar{6}_z$
	$\boldsymbol{C}_{3h}(\boldsymbol{C}_3)$	$3:\underline{m}$	$\bar{6}'$	$\bar{6}'$	$1, \pm 3_z, m'_z, \pm\bar{6}'_z$
	$\boldsymbol{C}_{6h}$	$6:m$	$6/m$	$\frac{6}{m}$	$1, \bar{1}, 2_z, m_z, \pm 3_z, \pm\bar{3}_z, \pm 6_z, \pm\bar{6}_z$
	$\boldsymbol{C}_{6h}(\boldsymbol{S}_6)$	$\underline{6}:\underline{m}$	$6'/m'$	$\frac{6'}{m'}$	$1, \bar{1}, \pm 3_z, \pm\bar{3}_z, 2'_z, m'_z, \pm 6'_z, \pm\bar{6}'_z$
	$\boldsymbol{C}_{6h}(\boldsymbol{C}_6)$	$6:\underline{m}$	$6/m'$	$\frac{6}{m'}$	$1, 2_z, \pm 3_z, \pm 6_z, \bar{1}', m'_z, \pm\bar{3}'_z, \pm\bar{6}'_z$
	$\boldsymbol{C}_{6h}(\boldsymbol{C}_{3h})$	$\underline{6}:m$	$6'/m$	$\frac{6'}{m}$	$1, m_z, \pm 3_z, \pm\bar{6}_z, \bar{1}', 2'_z, \pm\bar{3}'_z, \pm 6'_z$
	$\boldsymbol{D}_6$	$6:2$	622	622	$1, 6(2_\perp), 2_z, \pm 3_z, \pm 6_z$
	$\boldsymbol{D}_6(\boldsymbol{D}_3)$	$\underline{6}:2$	$6'22'$	$6'22'$	$1, 3(2_\perp), \pm 3_z, 3(2'_\perp), 2'_z, \pm 6'_z$
	$\boldsymbol{D}_6(\boldsymbol{C}_6)$	$6:\underline{2}$	$62'2'$	$62'2'$	$1, 2_z, \pm 3_z, \pm 6_z, 6(2'_\perp)$
	$\boldsymbol{C}_{6v}$	$6\cdot m$	$6mm$	$6mm$	$1, 2_z, 6(m_\perp), \pm 3_z, \pm 6_z$
	$\boldsymbol{C}_{6v}(\boldsymbol{C}_{3v})$	$\underline{6}\cdot m$	$6'mm'$	$6'mm'$	$1, 3(m_\perp), \pm 3_z, 2'_z, 3(m'_\perp), \pm 6'_z$
	$\boldsymbol{C}_{6v}(\boldsymbol{C}_6)$	$6\cdot\underline{m}$	$6m'm'$	$6m'm'$	$1, 2_z, \pm 3_z, \pm 6_z, 6(m'_\perp)$
	$\boldsymbol{D}_{3h}$	$m\cdot 3:m$	$\bar{6}m2$	$\bar{6}m2$	$1, 3(2_\perp), 3(m_\perp), m_z, \pm 3_z, \pm\bar{6}_z$
	$\boldsymbol{D}_{3h}(\boldsymbol{D}_3)$	$\underline{m}\cdot 3:\underline{m}$	$\bar{6}'2m'$	$\bar{6}'2m'$	$1, 3(2_\perp), \pm 3_z, 3(m'_\perp), m'_z, \pm\bar{6}'_z$
	$\boldsymbol{D}_{3h}(\boldsymbol{C}_{3v})$	$m\cdot 3:\underline{m}$	$\bar{6}'m2'$	$\bar{6}'m2'$	$1, 3(m_\perp), \pm 3_z, 3(2'_\perp), m'_z, \pm\bar{6}'_z$
	$\boldsymbol{D}_{3h}(\boldsymbol{C}_{3h})$	$\underline{m}\cdot 3:m$	$\bar{6}m'2'$	$\bar{6}m'2'$	$1, m_z, \pm 3_z, \pm\bar{6}_z, 3(2'_\perp), 3(m'_\perp)$
	$\boldsymbol{D}_{6h}$	$m\cdot 6:m$	$6/mmm$	$\frac{6}{m}\frac{2}{m}\frac{2}{m}$	$1, \bar{1}, 6(2_\perp), 2_z, 6(m_\perp), m_z, \pm 3_z, \pm\bar{3}_z, \pm 6_z, \pm\bar{6}_z$
	$\boldsymbol{D}_{6h}(\boldsymbol{D}_{3d})$	$m\cdot\underline{6}:\underline{m}$	$6'/m'mm'$	$\frac{6'}{m'}\frac{2}{m}\frac{2'}{m'}$	$1, \bar{1}, 3(2_\perp), 3(m_\perp), \pm 3_z, \pm\bar{3}_z, 3(2'_\perp), 2'_z, 3(m'_\perp), m'_z, \pm 6'_z, \pm\bar{6}'_z$
	$\boldsymbol{D}_{6h}(\boldsymbol{C}_{6h})$	$\underline{m}\cdot 6:m$	$6/mm'm'$	$\frac{6}{m}\frac{2'}{m'}\frac{2'}{m'}$	$1, \bar{1}, 2_z, m_z, \pm 3_z, \pm\bar{3}_z, \pm 6_z, \pm\bar{6}_z, 6(2'_\perp), 6(m'_\perp)$
	$\boldsymbol{D}_{6h}(\boldsymbol{D}_6)$	$\underline{m}\cdot 6:\underline{m}$	$6/m'm'm'$	$\frac{6}{m'}\frac{2}{m'}\frac{2}{m'}$	$1, 6(2_\perp), 2_z, \pm 3_z, \pm 6_z, \bar{1}', 6(m'_\perp), m'_z, \pm\bar{3}'_z, \pm\bar{6}'_z$
	$\boldsymbol{D}_{6h}(\boldsymbol{C}_{6v})$	$m\cdot 6:\underline{m}$	$6/m'mm$	$\frac{6}{m'}\frac{2'}{m}\frac{2'}{m}$	$1, 2_z, 6(m_\perp), \pm 3_z, \pm 6_z, \bar{1}', 6(2'_\perp), m'_z, \pm\bar{3}'_z, \pm\bar{6}'_z$
	$\boldsymbol{D}_{6h}(\boldsymbol{D}_{3h})$	$m\cdot\underline{6}:m$	$6'/mmm'$	$\frac{6'}{m}\frac{2'}{m}\frac{2}{m'}$	$1, 3(2_\perp), 3(m_\perp), m_z, \pm 3_z, \pm\bar{6}_z, \bar{1}', 3(2'_\perp), 2'_z, 3(m'_\perp), \pm\bar{3}'_z, \pm 6'_z$
Cubic	$\boldsymbol{T}$	$3/2$	23	23	$1, 3(2), 4(\pm 3)$
	$\boldsymbol{T}_h$	$\bar{6}/2$	$m\bar{3}$	$\frac{2}{m}\bar{3}$	$1, \bar{1}, 3(2), 3(m), 4(\pm 3), 4(\pm\bar{3})$
	$\boldsymbol{T}_h(\boldsymbol{T})$	$\underline{\bar{6}}/2$	$m'\bar{3}'$	$\frac{2}{m'}\bar{3}'$	$1, 3(2), 4(\pm 3), \bar{1}', 3(m'), 4(\pm\bar{3}')$
	$\boldsymbol{O}$	$3/4$	432	432	$1, 9(2), 4(\pm 3), 3(\pm 4)$
	$\boldsymbol{O}(\boldsymbol{T})$	$3/\underline{4}$	$4'32'$	$4'32'$	$1, 3(2), 4(\pm 3), 6(2'), 3(\pm 4')$
	$\boldsymbol{T}_d$	$3/\bar{4}$	$\bar{4}3m$	$\bar{4}3m$	$1, 3(2), 6(m), 4(\pm 3), 3(\pm\bar{4})$
	$\boldsymbol{T}_d(\boldsymbol{T})$	$3/\underline{\bar{4}}$	$\bar{4}'3m'$	$\bar{4}'3m'$	$1, 3(2), 4(\pm 3), 6(m'), 3(\pm\bar{4}')$
	$\boldsymbol{O}_h$	$\bar{6}/4$	$m\bar{3}m$	$\frac{4}{m}\bar{3}\frac{2}{m}$	$1, \bar{1}, 9(2), 9(m), 4(\pm 3), 4(\pm\bar{3}), 3(\pm 4), 3(\pm\bar{4})$
	$\boldsymbol{O}_h(\boldsymbol{T}_h)$	$\bar{6}/\underline{4}$	$m\bar{3}m'$	$\frac{4'}{m}\bar{3}\frac{2'}{m'}$	$1, \bar{1}, 3(2), 3(m), 4(\pm 3), 4(\pm\bar{3}), 6(2'), 6(m'), 3(\pm 4'), 3(\pm\bar{4}')$
	$\boldsymbol{O}_h(\boldsymbol{O})$	$\underline{\bar{6}}/4$	$m'\bar{3}'m'$	$\frac{4}{m'}\bar{3}'\frac{2}{m'}$	$1, 9(2), 4(\pm 3), 3(\pm 4), \bar{1}', 9(m'), 4(\pm\bar{3}'), 3(\pm\bar{4}')$
	$\boldsymbol{O}_h(\boldsymbol{T}_d)$	$\underline{\bar{6}}/\underline{4}$	$m'\bar{3}'m$	$\frac{4'}{m'}\bar{3}'\frac{2'}{m}$	$1, 3(2), 6(m), 4(\pm 3), 3(\pm\bar{4}), \bar{1}', 6(2'), 3(m'), 4(\pm\bar{3}'), 3(\pm 4')$

subgroup the elements of which are not multiplied by R. This notation is often used in the physics literature. In the crystallographic literature, the magnetic groups are defined by Hermann–Mauguin or Shubnikov symbols. In this type of designation, the symbols of elements multiplied by R are primed or underlined. The primed symbols are used in most of the recent publications. The Hermann–Mauguin and Shubnikov definitions differ slightly, as in the case of crystallographic groups. In Table 1.5.2.1, different symbols of magnetic point groups (trivial and nontrivial ones) are compared. This is done for the family that belongs to the crystallographic point group $\boldsymbol{D}_4 = 422$. The symbols of the symmetry elements of these four magnetic point groups are compared in Table 1.5.2.2.

Table 1.5.2.3 gives a list of the 90 magnetic point groups belonging to types 2 and 3. The Schoenflies, Shubnikov and Hermann–Mauguin symbols of the point groups are given in the table. The entries in the Hermann–Mauguin symbol refer to symmetry directions, as explained in Section 2.2.4 of *International Tables for Crystallography*, Vol. A (2002). The elements of symmetry of each point group are displayed using the Hermann–Mauguin symbols. The symbol $N(2_\perp)$ denotes N 180° rotations with axes perpendicular to the principal symmetry axis; $N(m_\perp)$ denotes N mirror planes with normals perpendicular to the principal symmetry axis. Similar definitions hold for the primed symbols $N(2'_\perp)$ and $N(m'_\perp)$. The point groups are arranged in families. The part of the Schoenflies symbol before the bracket is

Table 1.5.2.4. *List of the magnetic classes in which ferromagnetism is admitted*

(*a*) Triclinic

Symbol of symmetry class		Allowed direction of $\mathbf{M}_s$
Schoenflies	Hermann–Mauguin	
$\boldsymbol{C}_1$	1	Any
$\boldsymbol{C}_i$	$\bar{1}$	Any

(*b*) Monoclinic

Symbol of symmetry class		Allowed direction of $\mathbf{M}_s$
Schoenflies	Hermann–Mauguin	
$\boldsymbol{C}_2$	2	$\parallel 2$
$\boldsymbol{C}_2(\boldsymbol{C}_1)$	$2'$	$\perp 2'$
$\boldsymbol{C}_s = \boldsymbol{C}_{1h}$	m	$\perp m$
$\boldsymbol{C}_s(\boldsymbol{C}_1)$	m'	$\parallel m'$
$\boldsymbol{C}_{2h}$	$2/m$	$\parallel 2$
$\boldsymbol{C}_{2h}(\boldsymbol{C}_i)$	$2'/m'$	$\parallel m'$

(*c*) Orthorhombic

Symbol of symmetry class		Allowed direction of $\mathbf{M}_s$
Schoenflies	Hermann–Mauguin	
$\boldsymbol{D}_2(\boldsymbol{C}_2)$	$22'2'$	$\parallel 2$
$\boldsymbol{C}_{2v}(\boldsymbol{C}_2)$	$m'm'2$	$\parallel 2$
$\boldsymbol{C}_{2v}(\boldsymbol{C}_s)$	$m'm2'$	$\perp m$
$\boldsymbol{D}_{2h}(\boldsymbol{C}_{2h})$	$mm'm'$	$\perp m$

(*d*) Tetragonal

Symbol of symmetry class		Allowed direction of $\mathbf{M}_s$
Schoenflies	Hermann–Mauguin	
$\boldsymbol{C}_4$	4	$\parallel 4$
$\boldsymbol{S}_4$	$\bar{4}$	$\parallel \bar{4}$
$\boldsymbol{C}_{4h}$	$4/m$	$\parallel 4$
$\boldsymbol{D}_4(\boldsymbol{C}_4)$	$42'2'$	$\parallel 4$
$\boldsymbol{C}_{4v}(\boldsymbol{C}_4)$	$4m'm'$	$\parallel 4$
$\boldsymbol{D}_{2d}(\boldsymbol{S}_4)$	$\bar{4}2'm'$	$\parallel \bar{4}$
$\boldsymbol{D}_{4h}(\boldsymbol{C}_{4h})$	$4/mm'm'$	$\parallel 4$

(*e*) Trigonal

Symbol of symmetry class		Allowed direction of $\mathbf{M}_s$
Schoenflies	Hermann–Mauguin	
$\boldsymbol{C}_3$	3	$\parallel 3$
$\boldsymbol{S}_6$	$\bar{3}$	$\parallel 3$
$\boldsymbol{D}_3(\boldsymbol{C}_3)$	$32'$	$\parallel 3$
$\boldsymbol{C}_{3v}(\boldsymbol{C}_3)$	$3m'$	$\parallel 3$
$\boldsymbol{D}_{3d}(\boldsymbol{S}_6)$	$\bar{3}m'$	$\parallel 3$

(*f*) Hexagonal

Symbol of symmetry class		Allowed direction of $\mathbf{M}_s$
Schoenflies	Hermann–Mauguin	
$\boldsymbol{C}_6$	6	$\parallel 6$
$\boldsymbol{C}_{3h}$	$\bar{6}$	$\parallel \bar{6}$
$\boldsymbol{C}_{6h}$	$6/m$	$\parallel 6$
$\boldsymbol{D}_6(\boldsymbol{C}_6)$	$62'2'$	$\parallel 6$
$\boldsymbol{C}_{6v}(\boldsymbol{C}_6)$	$6m'm'$	$\parallel 6$
$\boldsymbol{D}_{3h}(\boldsymbol{C}_{3h})$	$\bar{6}m'2'$	$\parallel \bar{6}$
$\boldsymbol{D}_{6h}(\boldsymbol{C}_{6h})$	$6/mm'm'$	$\parallel 6$

the same for each member of a family. Each family begins with a trivial magnetic point group. It contains the same elements as the corresponding crystallographic point group; its Schoenflies symbol contains no brackets. For each nontrivial point group, the list of the elements of symmetry begins with the non-primed elements, which belong to a subgroup $\mathcal{H}$ of the head of the family $\mathcal{G}$. The number of the primed elements is equal to the number of non-primed ones and the total number of the elements is the same for all point groups of one family.

The overall number of the magnetic point groups of all three types is 122. There are two general statements concerning the magnetic point groups. The element $RC_3 = 3'$ does not appear in any of the magnetic point groups of type 3. Only trivial magnetic point groups (of both first and second type) belong to the families containing the point groups $\boldsymbol{C}_1 = 1$, $\boldsymbol{C}_3 = 3$ and $\boldsymbol{T} = 23$.

Only 31 magnetic point groups allow ferromagnetism. The different types of ferromagnetism (one-sublattice ferromagnet, ferrimagnet, weak ferromagnet, any magnetic order with nonzero magnetization) cannot be distinguished by their magnetic symmetry. Ferromagnetism is not admitted in any point group of type 1. For the magnetic point groups of the second type, ferromagnetism is not allowed if the point group contains more than one symmetry axis, more than one mirror plane or a mirror plane that is parallel to the axis. The same restrictions are valid for the point groups of type 3 (if the corresponding elements are not multiplied by R). If the point group contains $\bar{1}'$, ferromagnetic order is also forbidden. There are the following rules for the orientation of the axial vector of ferromagnetic magnetization $\mathbf{M}$: $\mathbf{M} \parallel N$, $\mathbf{M} \perp 2'$, $\mathbf{M} \perp m$, $\mathbf{M} \parallel m'$. Table 1.5.2.4 lists those magnetic point groups that admit ferromagnetic order (Tavger, 1958). The allowed direction of the magnetization vector is given for every point group. Ferromagnetic order is allowed in 13 point groups of the second type and 18 point groups of the third type.

All 31 point groups of magnetic symmetry allowing ferromagnetism are subgroups of the infinite noncrystallographic group

$$\boldsymbol{D}_{\infty h}(\boldsymbol{C}_{\infty h}) = \frac{\infty}{m}\frac{2'}{m'}.$$

The transition from a paramagnetic to a ferromagnetic state is always accompanied by a change of the magnetic symmetry.

1.5.2.2. *Magnetic lattices*

If the point group of symmetry describes the macroscopic properties of a crystal, its microscopic structure is determined by the space group, which contains the group of translations $\mathcal{T}$ as a subgroup. The elements $\mathbf{t}$ of $\mathcal{T}$ are defined by the following relation:

$$\mathbf{t} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3, \qquad (1.5.2.3)$$

where $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are basic primitive translation vectors and n_1, n_2, n_3 are arbitrary integers. The set of points $\mathbf{r}'$ obtained by applying all the translations of the group $\mathcal{T}$ to any point $\mathbf{r}$ defines a lattice. All sites of the crystallographic lattice are equivalent.

The structure of the ordered magnetics is described by the magnetic lattices and corresponding magnetic translation groups $\mathcal{M}_T$. In the magnetic translation groups $\mathcal{M}_T$, some of the elements $\mathbf{t}$ may be multiplied by R (we shall call them primed translations). The magnetic lattices then have two types of sites, which are not equivalent. One set is obtained by non-primed translations and the other set by the primed ones. The magnetic translation group $\mathcal{M}_T$ is isometric to the crystallographic one $\mathcal{G}_0$ that is obtained by replacing R by E in $\mathcal{M}_T$.

There are trivial magnetic translation groups, in which none of the translation elements is multiplied by R. The magnetic lattices of these groups coincide with crystallographic lattices.

Nontrivial magnetic translation groups can be constructed in analogy to relation (1.5.2.2). Zamorzaev (1957) showed that every translation group $\mathcal{T}$ has seven subgroups of index 2. If the basic primitive translations of the group $\mathcal{T}$ are $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, then the basic primitive translations of the seven subgroups $\mathcal{H}$ can be chosen as follows (see also Opechowski & Guccione, 1965)

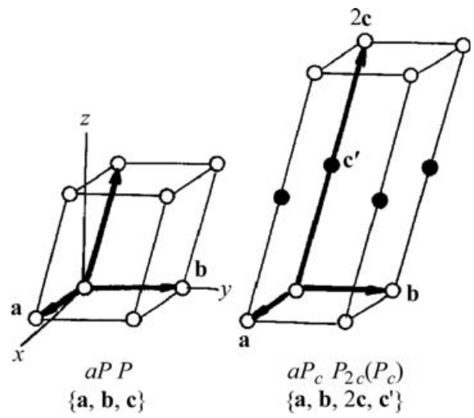

Fig. 1.5.2.1. Magnetic lattices of the triclinic system.

$$\mathcal{H}_1 : 2\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \qquad (1.5.2.4)$$

$$\mathcal{H}_2 : \mathbf{a}_1, 2\mathbf{a}_2, \mathbf{a}_3 \qquad (1.5.2.5)$$

$$\mathcal{H}_3 : \mathbf{a}_1, \mathbf{a}_2, 2\mathbf{a}_3 \qquad (1.5.2.6)$$

$$\mathcal{H}_4 : 2\mathbf{a}_1, \mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_3 \qquad (1.5.2.7)$$

$$\mathcal{H}_5 : 2\mathbf{a}_2, \mathbf{a}_2 + \mathbf{a}_3, \mathbf{a}_1 \qquad (1.5.2.8)$$

$$\mathcal{H}_6 : 2\mathbf{a}_3, \mathbf{a}_3 + \mathbf{a}_1, \mathbf{a}_2 \qquad (1.5.2.9)$$

$$\mathcal{H}_7 : 2\mathbf{a}_1, \mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_1 + \mathbf{a}_3. \qquad (1.5.2.10)$$

As an example, let us consider the case (1.5.2.5). In this case, the subgroup $\mathcal{H}$ consists of the following translations:

$$\mathbf{t}(\mathcal{H}) = n_1\mathbf{a}_1 + 2n_2\mathbf{a}_2 + n_3\mathbf{a}_3. \qquad (1.5.2.11)$$

Therefore the elements G_i of $(\mathcal{T} - \mathcal{H})$ [which corresponds to $(\mathcal{P} - \mathcal{H})$ in relation (1.5.2.2)] must have the following form:

$$\mathbf{t}(G_i) = n_1\mathbf{a}_1 + (2n_2 + 1)\mathbf{a}_2 + n_3\mathbf{a}_3. \qquad (1.5.2.12)$$

The corresponding magnetic translation group consists of the elements (1.5.2.12) multiplied by R and the elements (1.5.2.11).

The crystallographic lattices are classified into Bravais types or Bravais lattices. The magnetic lattices are classified into Bravais types of magnetic lattices. It turns out that there are 22 nontrivial magnetic Bravais types. Together with the trivial ones, there are 36 magnetic Bravais lattices.

Two types of smallest translation-invariant cells are in common use for the description of magnetically ordered structures: the crystallographic cell obtained if the magnetic order is neglected and the magnetic cell, which takes the magnetic order into account. The list of the basic translations of all the magnetic Bravais lattices was given by Zamorzaev (1957). The diagrams of the magnetic unit cells were obtained by Belov *et al.* (1957).

In Figs. 1.5.2.1–1.5.2.7, the diagrams of the magnetic unit cells of all 36 Bravais types are sketched in such a way that it is clear to which family the given cell belongs. All the cells of one family are displayed in one row. Such a row begins with the cell of the trivial magnetic lattice. All nontrivial cells of a family change into the trivial one of this family if R is replaced by E (to draw these diagrams we used those published by Opechowski & Guccione, 1965). Open and full circles are used to show the primed and unprimed translations. A line connecting two circles of the same type is an unprimed translation; a line connecting two circles of different types is a primed translation. The arrows in the trivial magnetic cell represent the primitive (primed or unprimed) translations for all the magnetic lattices of the family. The arrows in the nontrivial cells are primitive translations of the magnetic unit cell. The magnetic unit cell of a nontrivial magnetic lattice is generated by unprimed translations only. Its volume is twice the volume of the smallest cell generated by all (primed and unprimed) translations. The reason for this is that one of the primitive translations of the magnetic cell is twice a primitive primed translation. The crystallographic cell of many simple collinear or weakly non-collinear structures coincides with the smallest cell generated by the primed and unprimed translations. However, there are also magnetic structures with more complicated transformations from the crystallographic to the magnetic unit cell. The second line after each part of Figs. 1.5.2.1–1.5.2.7 gives, between braces, an extended vector basis of the magnetic translation group (Shubnikov & Koptsik, 1972). The first line

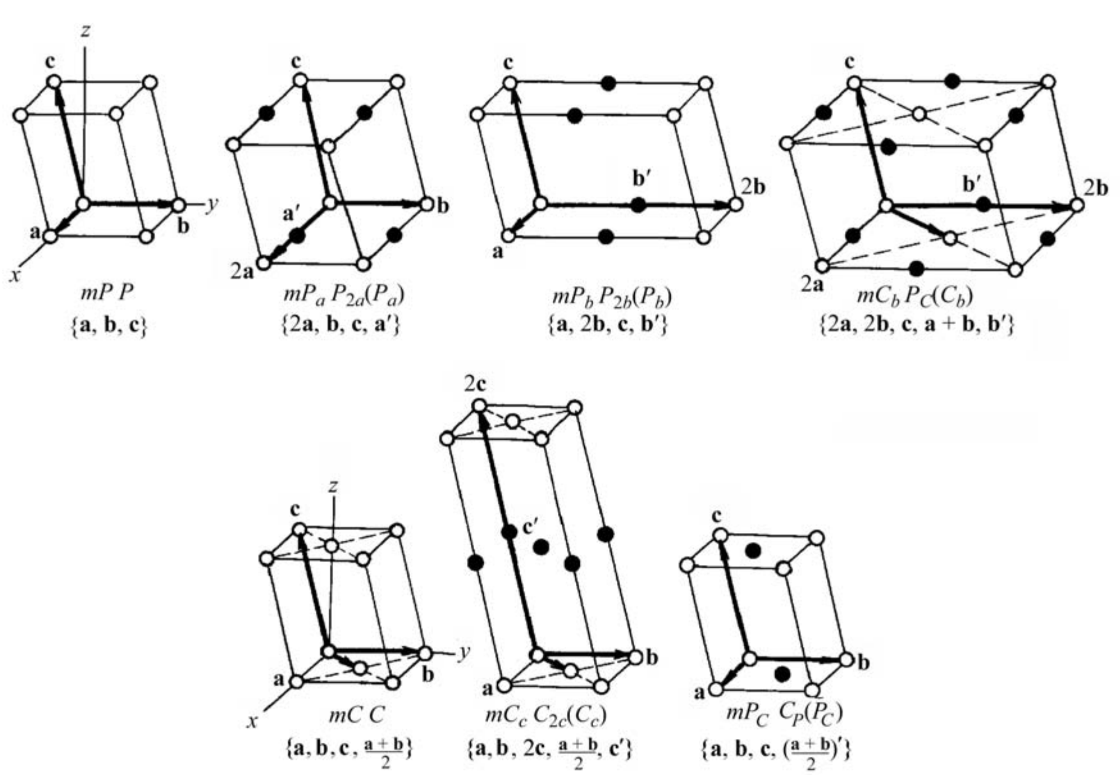

Fig. 1.5.2.2. Magnetic lattices of the monoclinic system (the y axis is the twofold axis).

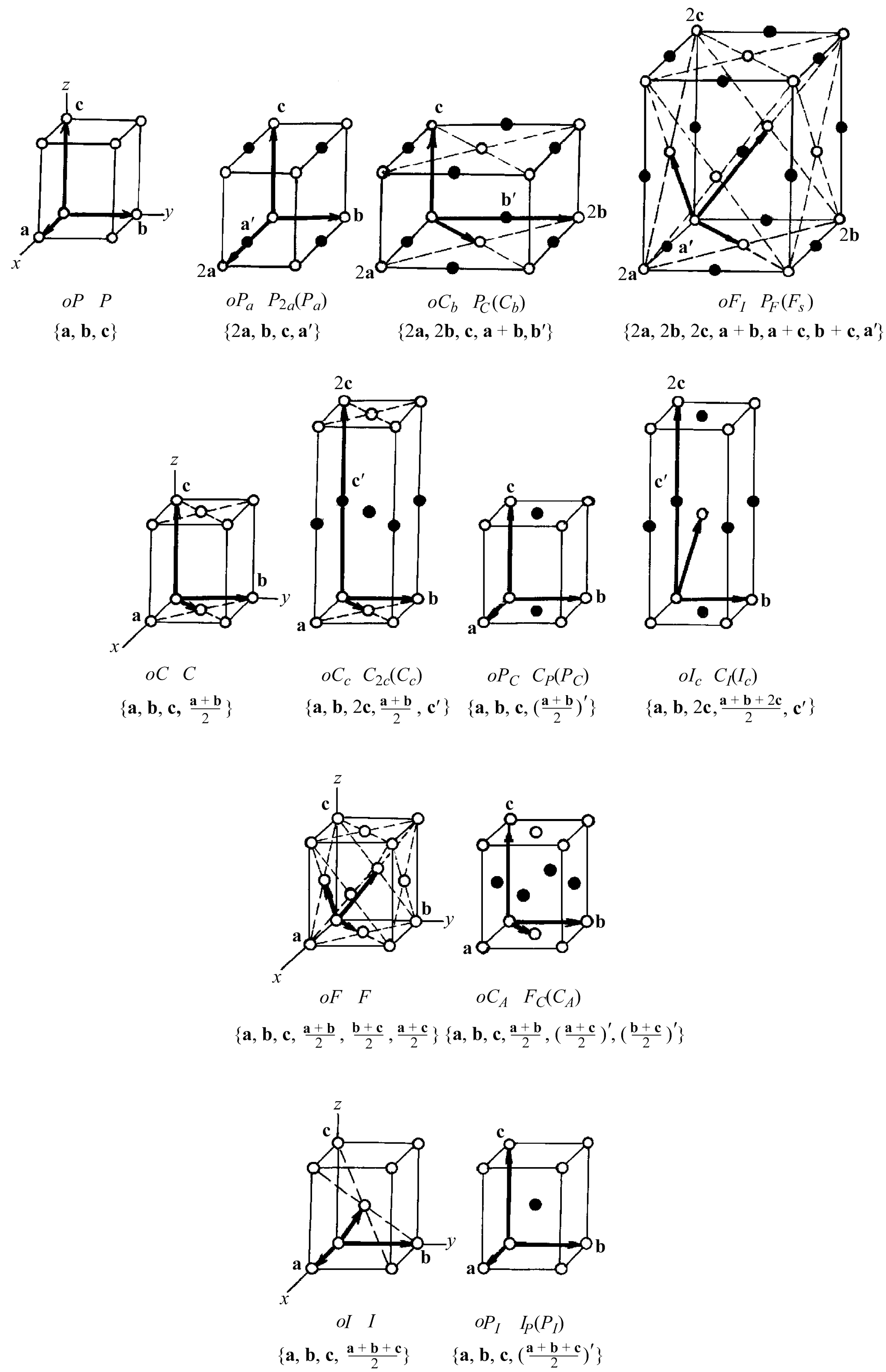

Fig. 1.5.2.3. Magnetic lattices of the orthorhombic system.

gives two symbols for each Bravais type: the symbol to the right was introduced by Opechowski & Guccione (1965). The symbol to the left starts with a lower-case letter giving the crystal system followed by a capital letter giving the centring type of the cell defined by the unprimed translations (*P*: primitive; *C*, *A*, *B*: *C*-, *A*-, *B*-centred; *I*: body-centred; *F*: all-face-centred). The subscript, which appears for the nontrivial Bravais types, indicates the translations that are multiplied by time inversion *R*.

Ferromagnetism is allowed only in trivial magnetic Bravais lattices. All nontrivial magnetic lattices represent antiferromagnetic order. There are only two magnetic sublattices in the simplest antiferromagnetic structures; one sublattice consists

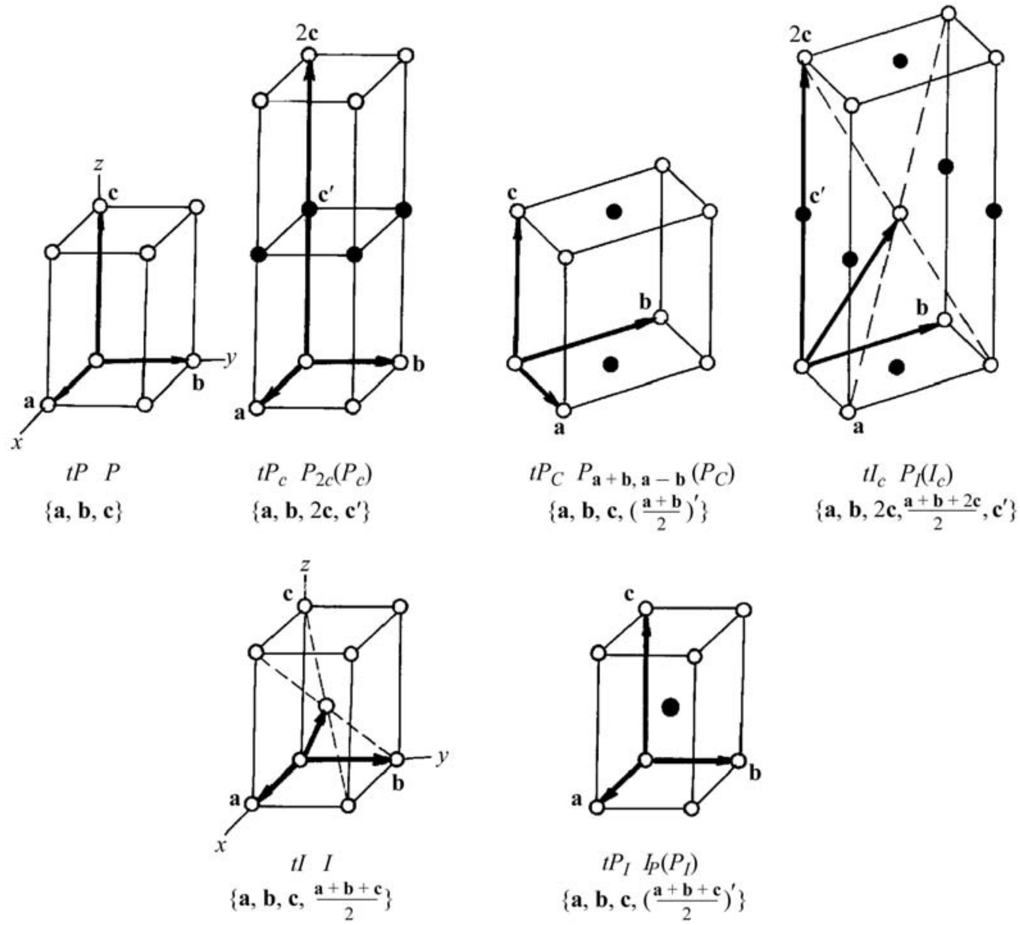

Fig. 1.5.2.4. Magnetic lattices of the tetragonal system.

of the magnetic ions located in the black sites and the other of the ions located in the white sites. All the magnetic moments of one sublattice are oriented in one direction and those of the other sublattice in the opposite direction. However, antiferromagnetism is allowed also in trivial lattices if the (trivial) magnetic cell contains more than one magnetic ion. The magnetic point group must be nontrivial in this case. The situation is more complicated in case of strongly non-collinear structures. In such structures (triangle, 90° *etc.*), the magnetic lattice can differ from the crystallographic one despite the fact that none of the translations is multiplied by *R*. The magnetic elementary cell will possess three or four magnetic ions although the crystallographic cell possesses only one. An example of such a situation is shown in Fig. 1.5.1.3(*c*). More complicated structures in which the magnetic lattice is incommensurate with the crystallographic one also exist. We shall not discuss the problems of such systems in this chapter.

1.5.2.3. *Magnetic space groups*

There are 1651 magnetic space groups $\mathcal{M}_G$, which can be divided into three types. Type I, $\mathcal{M}_{G1}$, consists of the 230 crystallographic space groups to which *R* is added. Crystals belonging to these trivial magnetic space groups show no magnetic order; they are para- or diamagnetic.

Type II, $\mathcal{M}_{G2}$, consists of the same 230 crystallographic groups which do not include *R* in any form. In the ordered magnetics, which belong to the magnetic space groups of this type, the

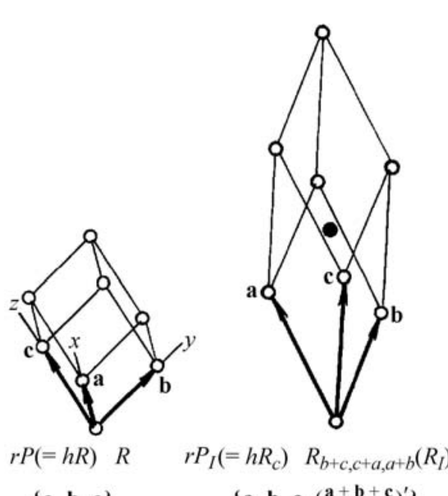

Fig. 1.5.2.5. Magnetic lattices of the rhombohedral system.

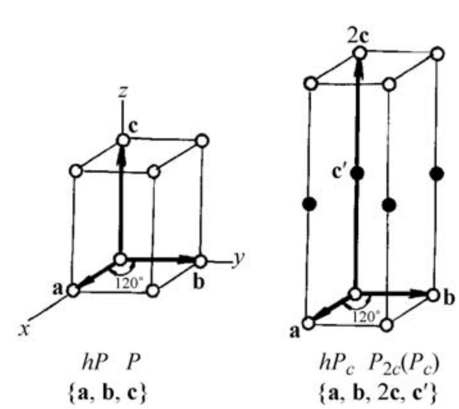

Fig. 1.5.2.6. Magnetic lattices of the hexagonal system.

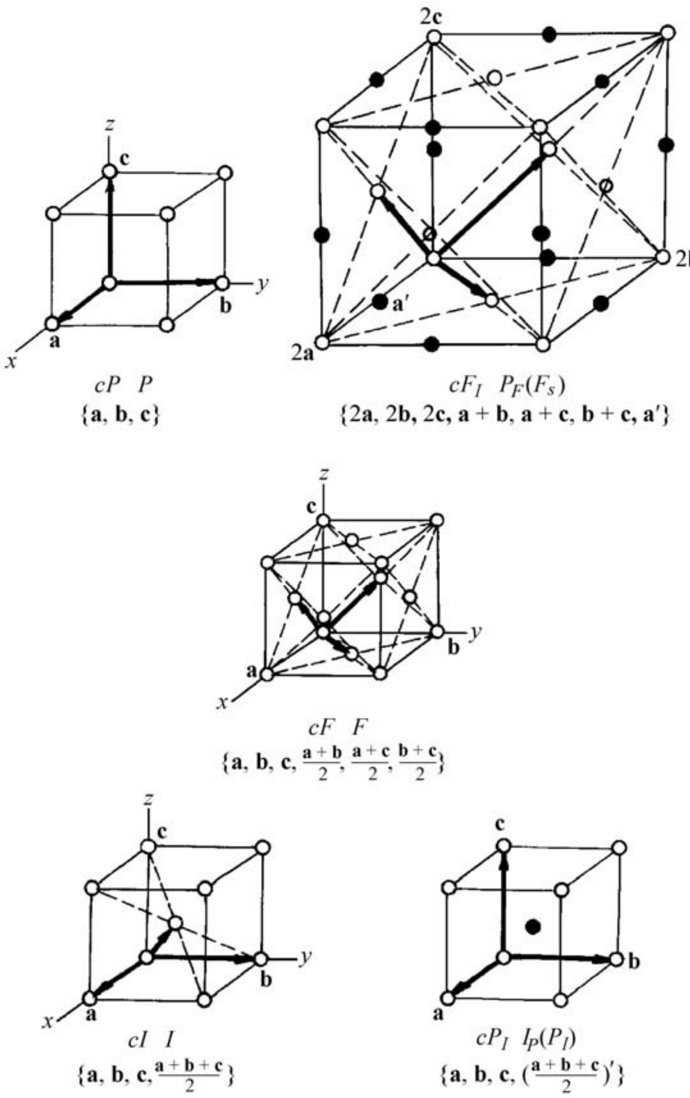

Fig. 1.5.2.7. Magnetic lattices of the cubic system.

magnetic unit cell coincides with the classical one. Forty-four groups of type II describe different ferromagnetic crystals; the remaining antiferromagnets.

The nontrivial magnetic space groups belong to type III, $\mathcal{M}_{G3}$. This consists of 1191 groups, in which R enters only in combination with rotations, reflections or translations. These groups have the structure described by relation (1.5.2.2). The magnetic space groups of this type are divided into two subtypes.

Subtype III^a contains those magnetic space groups $\mathcal{M}_{G3}$ in which R is not combined with translations. In these groups, the magnetic translation group is trivial. To these space groups correspond magnetic point groups of type $\mathcal{M}_{P3}$. There are 674 magnetic space groups of subtype III^a; 231 of them admit ferromagnetism, the remaining 443 describe antiferromagnets.

In the magnetic space groups of the subtype III^b, R is combined with translations and the corresponding point groups are of type $\mathcal{M}_{P1}$. They have a nontrivial magnetic Bravais lattice. There are 517 magnetic space groups of this subtype; they describe antiferromagnets.

In summary, the 230 magnetic space groups that describe dia- and paramagnets are of type I, the 275 that admit spontaneous magnetization are of types II and III^a; the remaining 1146 magnetic space groups (types II, III^a and III^b) describe antiferromagnets.

1.5.2.4. *Exchange symmetry*

The classification of magnetic structures on the basis of the magnetic (point and space) groups is an exact classification. However, it neglects the fundamental role of the exchange energy, which is responsible for the magnetic order (see Sections 1.5.1.2 and 1.5.3.2). To describe the symmetry of the magnetically ordered crystals only by the magnetic space groups means the loss of significant information concerning those properties of these materials that are connected with the higher symmetry of the exchange forces. Andreev & Marchenko (1976, 1980) have introduced the concept of exchange symmetry.

The exchange forces do not depend on the directions of the spins (magnetic moments) of the ions relative to the crystallographic axes and planes. They depend only on the relative directions of the spins. Thus the exchange group $\mathcal{G}_{ex}$ contains an infinite number of rotations U of spin space, *i.e.* rotations of all the spins (magnetic moments) through the same angle about the same axis. The components of the magnetic moment density $\mathbf{m}(\mathbf{r})$ transform like scalars under all rotations of spin space. The exchange symmetry group $\mathcal{G}_{ex}$ contains those combinations of the space transformation elements, the rotations U of spin space and the element R with respect to which the values $m(\mathbf{r})$ are invariant. Setting all the elements U and R equal to the identity transformation, we obtain one of the ordinary crystallographic space groups $\mathcal{G}$. This space group defines the symmetry of the charge density $\rho(\mathbf{r})$ and of all the magnetic scalars in the crystal. However, the vectors $\mathbf{m}(\mathbf{r})$ may not be invariant with respect to $\mathcal{G}$.

The concept of exchange symmetry makes it possible to classify all the magnetic structures (including the incommensurate ones) with the help of not more than three orthogonal magnetic vectors. We shall discuss this in more detail in Section 1.5.3.3.

More information about magnetic symmetry can be found in Birss (1964), Cracknell (1975), Joshua (1991), Koptsik (1966), Landau & Lifshitz (1957), Opechowski & Guccione (1965), and in Sirotin & Shaskol'skaya (1979).

1.5.3. Phase transitions into a magnetically ordered state

Most transitions from a paramagnetic into an ordered magnetic state are second-order phase transitions. A crystal with a given crystallographic symmetry can undergo transitions to different ordered states with different magnetic symmetry. In Section 1.5.3.3, we shall give a short review of the theory of magnetic second-order phase transitions. As was shown by Landau (1937), such a transition causes a change in the magnetic symmetry. The magnetic symmetry group of the ordered state is a subgroup of the magnetic group of the material in the paramagnetic state. But first we shall give a simple qualitative analysis of such transitions.

To find out what ordered magnetic structures may be obtained in a given material and to which magnetic group they belong, one has to start by considering the crystallographic space group $\mathcal{G}$ of the crystal under consideration. It is obvious that a crystal in which the unit cell contains only one magnetic ion can change only into a ferromagnetic state if the magnetic unit cell of the ordered state coincides with the crystallographic one. If a transition into an antiferromagnetic state occurs, then the magnetic cell in the ordered state will be larger than the crystallographic one if the latter contains only one magnetic ion. Such antiferromagnets usually belong to the subtype III^b described in Section 1.5.2.3. In Section 1.5.3.1, we shall consider crystals that transform into an antiferromagnetic state without change of the unit cell. This is possible only if the unit cell possesses two or more magnetic ions. To find the possible magnetic structures in this case, one has to consider those elements of symmetry which interchange the positions of the ions inside the unit cell (especially glide planes and rotation axes). Some of these elements displace the magnetic ion without changing its magnetic moment, and others change the moment of the ion. It is also essential to know the positions of all these elements in the unit cell. All this information is contained in the space group $\mathcal{G}$. If the magnetic ordering occurs without change of the unit cell, the translation