The refractive indices $n^{\pm}(\omega, \theta, \varphi) = [\varepsilon^{\pm}(\omega, \theta, \varphi)]^{1/2}$, $(n^+ > n^-)$, real in the case of a lossless medium, are the two solutions of the Fresnel equation (Yao & Fahlen, 1984):

$$\begin{aligned} n^{\pm} &= \left[\frac{2}{-B \mp (B^2 - 4C)^{1/2}}\right]^{1/2} \\ B &= -u_x^2(b+c) - u_y^2(a+c) - u_z^2(a+b) \\ C &= u_x^2 bc + u_y^2 ac + u_z^2 ab \\ a &= n_x^{-2}(\omega), \quad b = n_y^{-2}(\omega), \quad c = n_z^{-2}(\omega). \end{aligned} \tag{1.7.3.6}$$

$n_x(\omega)$, $n_y(\omega)$ and $n_z(\omega)$ are the principal refractive indices of the index ellipsoid at the circular frequency ω.

Equation (1.7.3.6) describes a double-sheeted three-dimensional surface: for a direction of propagation $\mathbf{u}$ the distances from the origin of the optical frame to the sheets (+) and (−) correspond to the roots n^+ and n^-. This surface is called the index surface or the wavevector surface. The quantity $(n^+ - n^-)$ or $(n^- - n^+)$ is the birefringency. The waves (+) and (−) have the phase velocities c/n^+ and c/n^-, respectively.

Equation (1.7.3.6) and its dispersion in frequency are often used in nonlinear optics, in particular for the calculation of the phase-matching directions which will be defined later. In the regions of transparency of the crystal, the frequency law is well described by a Sellmeier equation, which is the case for normal dispersion where the refractive indices increase with frequency (Hadni, 1967):

$$n^{\pm}(\omega_i) < n^{\pm}(\omega_j) \quad \text{for} \quad \omega_i < \omega_j. \tag{1.7.3.7}$$

If ω_i or ω_j are near an absorption peak, even weak, $n^{\pm}(\omega_i)$ can be greater than $n^{\pm}(\omega_j)$; this is called abnormal dispersion.

The dielectric displacements $\mathbf{D}^{\pm}$, the electric fields $\mathbf{E}^{\pm}$, the energy flux given by the Poynting vector $\mathbf{S}^{\pm} = \mathbf{E}^{\pm} \times \mathbf{H}^{\pm}$ and the collinear wavevectors $\mathbf{k}^{\pm}$ are coplanar and define the orthogonal vibration planes $\Pi^{\pm}$ (Shuvalov, 1981). Because of anisotropy, $\mathbf{k}^{\pm}$ and $\mathbf{S}^{\pm}$, and hence $\mathbf{D}^{\pm}$ and $\mathbf{E}^{\pm}$, are non-collinear in the general case as shown in Fig. 1.7.3.1: the walk-off angles, also termed double-refraction angles, $\rho^{\pm} = \arccos(\mathbf{d}^{\pm} \cdot \mathbf{e}^{\pm}) = \arccos(\mathbf{u} \cdot \mathbf{s}^{\pm})$ are different in the general case; $\mathbf{d}^{\pm}$, $\mathbf{e}^{\pm}$, $\mathbf{u}$ and $\mathbf{s}^{\pm}$ are the unit vectors associated with $\mathbf{D}^{\pm}$, $\mathbf{E}^{\pm}$, $\mathbf{k}^{\pm}$ and $\mathbf{S}^{\pm}$, respectively. We shall see later that the efficiency of a nonlinear interaction is strongly conditioned by $\mathbf{k}$, $\mathbf{E}$ and ρ, which only depend on $\chi^{(1)}(\omega)$, that is to say on the linear optical properties.

The directions $\mathbf{S}^+$ and $\mathbf{S}^-$ are the directions normal to the sheets (+) and (−) of the index surface at the points n^+ and n^-.

For a plane wave, the time-average Poynting vector is (Yariv & Yeh, 2002)

$$\begin{aligned} \|\mathbf{S}^{\pm}(\omega)\| &= \left\|\tfrac{1}{2} Re\left[\mathbf{E}^{\pm}(\omega) \times \mathbf{H}^{\pm *}(\omega)\right]\right\| \\ &= \tfrac{1}{2} \frac{\|\mathbf{k}^{\pm}(\omega)\|}{\mu_0 \omega} \left\|\mathbf{E}^{\pm}(\omega)\right\|^2 \cos^2 \rho^{\pm}(\omega). \end{aligned} \tag{1.7.3.8}$$

$\|\mathbf{S}^{\pm}\|$ is the energy flow $I = \hbar\omega N^{\pm}$, which is a power per unit area *i.e.* the intensity, where $\hbar\omega$ is the energy of the photon and $N^{\pm}$ are the photons flows. $\rho^{\pm}(\omega)$ is the angle between $\mathbf{S}^{\pm}$ and $\mathbf{u}$; it is detailed later on.

The unit electric field vectors $\mathbf{e}^+$ and $\mathbf{e}^-$ are calculated from the propagation equation projected on the three axes of the optical

Table 1.7.2.4. *Nonzero $\chi^{(3)}$ coefficients and equalities between them in the general case*

Symmetry class	$\chi^{(3)}$ nonzero elements
Triclinic C_1 (1), C_i ($\bar{1}$)	All 81 elements are independent and nonzero
Monoclinic C_s (*m*), C_2 (2), C_{2h} ($\frac{2}{m}$) (twofold axis parallel to *z*)	*xxxx, xyyy, xyzz, xzyz, xzzy, xxzz, xzxz, xzzx, xxyy, xyxy, xyyx, xxxy, xxyx, xyxx, yxxx, yyyy, yyzz, yzyz, yzzy, yxzz, yzxz, yzzx, yxyy, yyxy, yyyx, yxxy, yxyx, yyxx, zzzz, zyyz, zyzy, zzyy, zxxz, zxzx, zzxx, zxyz, zxzy, zyxz, zzxy, zyzx, zzyx*
Orthorhombic C_{2v} (*mm*2), D_2 (222), D_{2h} (*mmm*) (twofold axis parallel to *z*)	*xxxx, xxzz, xzxz, xzzx, xxyy, xyxy, xyyx, yyyy, yyzz, yzyz, yzzy, yxxy, yxyx, yyxx, zzzz, zyyz, zyzy, zzyy, zxxz, zxzx, zzxx*
Tetragonal S_4 ($\bar{4}$), C_4 (4), C_{4h} ($\frac{4}{m}$)	*xxxx = yyyy, xyyy = −yxxx, xyzz = −yxzz, xzyz = −yzxz, xzzy = −yzzx, xxzz = yyzz, xzxz = yzyz, xzzx = yzzy, xxyy = yyxx, xyxy = yxyx, xyyx = yxxy, xxxy = −yyyx, xxyx = −yyxy, xyxx = −yxyy, zzzz, zyyz = zxxz, zyzy = zxzx, zzyy = zzxx, zxyz = −zyxz, zxzy = −zyzx, zzxy = −zzyx*
C_{4v} (4*mm*), D_{2d} ($\bar{4}2m$), D_4 (422), D_{4h} ($\frac{4}{m}mm$)	*xxxx = yyyy, xxzz = yyzz, xzxz = yzyz, xzzx = yzzy, xxyy = yyxx, xyxy = yxyx, xyyx = yxxy, zzzz, zyyz = zxxz, zyzy = zxzx, zzyy = zzxx*
Hexagonal C_{3h} ($\bar{6}$), C_6 (6), C_{6h} ($\frac{6}{m}$)	*xxxx = yyyy = xxyy + xyxy + xyyx, xyyy = xxxy + xxyx + xyxx = −yxxx, xyzz = −yxzz, xzyz = −yzxz, xzzy = −yzzx, xxzz = yyzz, xzxz = yzyz, xzzx = yzzy, xxyy = yyxx, xyxy = yxyx, xyyx = yxxy, xxxy = −yyyx, xxyx = −yyxy, xyxx = −yxyy, zzzz, zyyz = zxxz, zyzy = zxzx, zzyy = zzxx, zxyz = −zyxz, zxzy = −zyzx, zzxy = −zzyx*
C_{6v} (6*mm*), D_{3h} ($\bar{6}2m$), D_6 (622), D_{6h} ($\frac{6}{m}mm$)	*xxxx = yyyy = xxyy + xyxy + xyyx, xxzz = yyzz, xzxz = yzyz, xzzx = yzzy, xxyy = yyxx, xyxy = yxyx, xyyx = yxxy, zzzz, zyyz = zxxz, zyzy = zxzx, zzyy = zzxx*
Trigonal C_3 (3), C_{3i} ($\bar{3}$)	*xxxx = yyyy = xxyy + xyxy + xyyx, xyyy = xxxy + xxyx + xyxx = −yxxx, xyzz = −yxzz, xzyz = −yzxz, xzzy = −yzzx, xyyz = yxyz = yyxz = −xxxz, xyzy = yyzx = yxzy = −xxzx, xzyy = yzxy = yzyx = −xzxx, xxzz = yyzz, xzxz = yzyz, xzzx = yzzy, xxyy = yyxx, xyxy = yxyx, xyyx = yxxy, xxxy = −yyyx, xxyx = −yyxy, xyxx = −yxyy, xxyz = xyxz = yxxz = −yyyz, xxzy = xyzx = yxzx = −yyzy, xzxy = xzyx = yzxx = −yzyy, −zxxx = zxyy = zyxy = zyyx, −zyyy = zxxy = zxyx = zyxx, zzzz, zyyz = zxxz, zyzy = zxzx, zzyy = zzxx, zxyz = −zyxz, zxzy = −zyzx, zzxy = −zzyx*
C_{3v} (3*m*), D_3 (32), D_{3d} ($\bar{3}m$) (mirror perpendicular to *x*) (twofold axis parallel to *x*)	*xxxx = yyyy = xxyy + xyxy + xyyx, xxzz = yyzz, xzxz = yzyz, xzzx = yzzy, xxyy = yyxx, xyxy = yxyx, xyyx = yxxy, xxyz = xyxz = yxxz = −yyyz, xxzy = xyzx = yxzx = −yyzy, xzxy = xzyx = yzxx = −yzyy, −zyyy = zxxy = zxyx = zyxx, zzzz, zyyz = zxxz, zyzy = zxzx, zzyy = zzxx*
Cubic *T* (23), T_h (*m*3)	*xxxx = yyyy = zzzz, xxzz = yyxx = zzyy, xzxz = yxyx = zyzy, xzzx = yxxy = zyyz, xxyy = yyzz = zzxx, xyxy = yzyz = zxzx, xyyx = yzzy = zxxz*
T_d ($\bar{4}3m$), *O* (432), O_h (*m*3*m*)	*xxxx = yyyy = zzzz, xxzz = xxyy = yyzz = yyxx = zzyy = zzxx, xzxz = xyxy = yzyz = yxyx = zyzy = zxzx, xzzx = xyyx = yzzy = yxxy = zyyz = zxxz*