The property of 'belong to the same left coset' is an equivalence relation, therefore two left cosets of the same subgroup are either identical or have no elements in common.

Proposition 3.2.3.4. The union of all distinct left cosets of F_1 in G constitutes a partition of G and is called the *decomposition of G into the left cosets of* F_1. If the set of left cosets of F_1 in G is finite, then the decomposition of G into the left cosets of F_1 can be expressed as

$$G = g_1F_1 \cup g_2F_1 \cup \ldots \cup g_nF_1 = \bigcup_{i=1}^{n} g_iF_1, \qquad (3.2.3.19)$$

where the symbol $\cup$ is the set-theoretical union (see Section 3.2.3.1). For the representative g_1 of the first left coset the unit element e is usually chosen, $g_1 = e$. Then the first left coset is identical with the subgroup F_1. The number of elements in each left coset of the decomposition is equal to the order of the group F_1.

The set of left-coset representatives $\{g_1, g_2, \ldots g_n\}$ is sometimes called a *left transversal to* F_1 *in* G.

The number n of distinct left cosets is called the *index of the subgroup* F_1 *in the group* G and is denoted by the symbol $[G : F_1]$. If the groups G and F_1 are of finite order then

$$n = [G : F_1] = |G| : |F_1|, \qquad (3.2.3.20)$$

where $|G|$ and $|F_1|$ are the orders of G and F_1, respectively. From this equation follows:

Lagrange's theorem: the order of a finite group is a multiple of the order of each of its subgroups. Alternatively, the index of a subgroup and the order of a finite subgroup are divisors of the group order.

We note that an infinite subgroup of an infinite group can have a finite index. Important examples are subgroups of translational groups of crystallographic space groups and subgroups of space groups (see Example [oC] 3.2.3.32 in Section 3.2.3.3.5).

The decompositions of crystallographic point groups into left and right cosets are available in the software *GI★KoBo*-1, under *Subgroups\View\Twinning Group*.

Proposition 3.2.3.5. Let L_1 be an intermediate group $F_1 \subset L_1 \subset G$. The group G can be decomposed into left cosets of L_1,

$$G = h_1L_1 \cup h_2L_1 \cup \ldots \cup h_mL_1 = \bigcup_{j=1}^{m} h_jL_1, \qquad (3.2.3.21)$$

where

$$m = [G : L_1] = |G| : |L_1|, \qquad (3.2.3.22)$$

and the group L_1 into left cosets of F_1,

$$L_1 = p_1F_1 \cup p_2F_1 \cup \ldots \cup p_dF_1 = \bigcup_{k=1}^{d} p_kF_1, \qquad (3.2.3.23)$$

where

$$d = [L_1 : F_1] = |L_1| : |F_1|. \qquad (3.2.3.24)$$

Then the decomposition of G into left cosets of F_1 can be written in the form

$$G = \bigcup_{j=1}^{m}\bigcup_{k=1}^{d} h_jp_kF_1 \qquad (3.2.3.25)$$

and the index n of F_1 in G can be expressed as a product of indices m and d,

$$n = [G : F_1] = [G : L_1][L_1 : F_1] = md. \qquad (3.2.3.26)$$

Decompositions (3.2.3.19) and (3.2.3.21) of a group into left cosets enable one to divide a set of objects into classes of symmetrically equivalent objects (see Section 3.2.3.3.4). The concept of domain states is based on this result (see Section 3.4.2).

3.2.3.2.4. *Conjugate subgroups*

Two subgroups F_i and F_k are *conjugate subgroups* if there exists an element g of G such that

$$gF_ig^{-1} = F_k, \quad g \in G. \qquad (3.2.3.27)$$

More explicitly, one says that the subgroup F_k is *conjugate by g* (or *conjugate under G*) *to the subgroup* F_i. Conjugate subgroups are isomorphic.

The property of 'being conjugate' is an equivalence relation. The set of all subgroups of a group G can therefore be partitioned into disjoint classes of conjugate subgroups. Conjugate subgroups of crystallographic point groups are given in Table 3.4.2.7 and in the software *GI★KoBo*-1, under *Subgroups\View\Twinning Group*.

3.2.3.2.5. *Normalizers*

The collection of all elements g that fulfil the relation

$$gF_ig^{-1} = F_i, \quad g \in G, \qquad (3.2.3.28)$$

constitutes a group denoted by $N_G(F_i)$ and is called the *normalizer of* F_i *in G. The normalizer* $N_G(F_i)$ *is a subgroup of G and a supergroup of* F_i,

$$F_i \subseteq N_G(F_i) \subseteq G. \qquad (3.2.3.29)$$

The normalizer $N_G(F_i)$ determines the subgroups conjugate to F_i under G (see Example 3.2.3.10). The number m of subgroups conjugate to a subgroup F_i under G equals the index of $N_G(F_i)$ in G:

$$m = [G : N_G(F_i)] = |G| : |N_G(F_i)|, \qquad (3.2.3.30)$$

where the last equation holds for finite G and F_i.

Normalizers of the subgroups of crystallographic point groups are available in Table 3.4.2.7 and in the software *GI★KoBo*-1 under *Subgroups\View\Twinning Group*.

3.2.3.2.6. *Normal subgroups*

Among subgroups of a group, a special role is played by normal subgroups. A subgroup H of G is a *normal* (*invariant*, *self-conjugate*) *subgroup* of G if and only if it fulfils any of the following conditions:

(1) The subgroup H of G has no conjugate subgroups under G. (No subscript is therefore needed in the symbol of a normal subgroup H.)

(2) The normalizer $N_G(H)$ of H equals the group G,

$$N_G(H) = G. \qquad (3.2.3.31)$$

(3) Every element g of G commutes with H, or, equivalently, each left coset gH equals the right coset Hg:

$$gH = Hg \text{ for every } g \in G. \qquad (3.2.3.32)$$

For a normal subgroup H of a group G a special symbol $\triangleleft$ is often used instead of $\subset$, $H \triangleleft G$.