

3.2. TWINNING AND DOMAIN STRUCTURES

The property of ‘belong to the same left coset’ is an equivalence relation, therefore two left cosets of the same subgroup are either identical or have no elements in common.

*Proposition 3.2.3.4.* The union of all distinct left cosets of  $F_1$  in  $G$  constitutes a partition of  $G$  and is called the *decomposition of  $G$  into the left cosets of  $F_1$* . If the set of left cosets of  $F_1$  in  $G$  is finite, then the decomposition of  $G$  into the left cosets of  $F_1$  can be expressed as

$$G = g_1F_1 \cup g_2F_1 \cup \dots \cup g_nF_1 = \bigcup_{i=1}^n g_iF_1, \quad (3.2.3.19)$$

where the symbol  $\cup$  is the set-theoretical union (see Section 3.2.3.1). For the representative  $g_1$  of the first left coset the unit element  $e$  is usually chosen,  $g_1 = e$ . Then the first left coset is identical with the subgroup  $F_1$ . The number of elements in each left coset of the decomposition is equal to the order of the group  $F_1$ .

The set of left-coset representatives  $\{g_1, g_2, \dots, g_n\}$  is sometimes called a *left transversal to  $F_1$  in  $G$* .

The number  $n$  of distinct left cosets is called the *index of the subgroup  $F_1$  in the group  $G$*  and is denoted by the symbol  $[G : F_1]$ . If the groups  $G$  and  $F_1$  are of finite order then

$$n = [G : F_1] = |G| : |F_1|, \quad (3.2.3.20)$$

where  $|G|$  and  $|F_1|$  are the orders of  $G$  and  $F_1$ , respectively. From this equation follows:

*Lagrange’s theorem:* the order of a finite group is a multiple of the order of each of its subgroups. Alternatively, the index of a subgroup and the order of a finite subgroup are divisors of the group order.

We note that an infinite subgroup of an infinite group can have a finite index. Important examples are subgroups of translational groups of crystallographic space groups and subgroups of space groups (see Example [oC] 3.2.3.32 in Section 3.2.3.3.5).

The decompositions of crystallographic point groups into left and right cosets are available in the software *GI★KoBo-1*, under *Subgroups\View\Twinning Group*.

*Proposition 3.2.3.5.* Let  $L_1$  be an intermediate group  $F_1 \subset L_1 \subset G$ . The group  $G$  can be decomposed into left cosets of  $L_1$ ,

$$G = h_1L_1 \cup h_2L_1 \cup \dots \cup h_mL_1 = \bigcup_{j=1}^m h_jL_1, \quad (3.2.3.21)$$

where

$$m = [G : L_1] = |G| : |L_1|, \quad (3.2.3.22)$$

and the group  $L_1$  into left cosets of  $F_1$ ,

$$L_1 = p_1F_1 \cup p_2F_1 \cup \dots \cup p_dF_1 = \bigcup_{k=1}^d p_kF_1, \quad (3.2.3.23)$$

where

$$d = [L_1 : F_1] = |L_1| : |F_1|. \quad (3.2.3.24)$$

Then the decomposition of  $G$  into left cosets of  $F_1$  can be written in the form

$$G = \bigcup_{j=1}^m \bigcup_{k=1}^d h_jp_kF_1 \quad (3.2.3.25)$$

and the index  $n$  of  $F_1$  in  $G$  can be expressed as a product of indices  $m$  and  $d$ ,

$$n = [G : F_1] = [G : L_1][L_1 : F_1] = md. \quad (3.2.3.26)$$

Decompositions (3.2.3.19) and (3.2.3.21) of a group into left cosets enable one to divide a set of objects into classes of symmetrically equivalent objects (see Section 3.2.3.3.4). The concept of domain states is based on this result (see Section 3.4.2).

3.2.3.2.4. Conjugate subgroups

Two subgroups  $F_i$  and  $F_k$  are *conjugate subgroups* if there exists an element  $g$  of  $G$  such that

$$gF_i g^{-1} = F_k, \quad g \in G. \quad (3.2.3.27)$$

More explicitly, one says that the subgroup  $F_k$  is *conjugate by  $g$  (or conjugate under  $G$ ) to the subgroup  $F_i$* . Conjugate subgroups are isomorphic.

The property of ‘being conjugate’ is an equivalence relation. The set of all subgroups of a group  $G$  can therefore be partitioned into disjoint classes of conjugate subgroups. Conjugate subgroups of crystallographic point groups are given in Table 3.4.2.7 and in the software *GI★KoBo-1*, under *Subgroups\View\Twinning Group*.

3.2.3.2.5. Normalizers

The collection of all elements  $g$  that fulfil the relation

$$gF_i g^{-1} = F_i, \quad g \in G, \quad (3.2.3.28)$$

constitutes a group denoted by  $N_G(F_i)$  and is called the *normalizer of  $F_i$  in  $G$* . The normalizer  $N_G(F_i)$  is a subgroup of  $G$  and a supergroup of  $F_i$ ,

$$F_i \subseteq N_G(F_i) \subseteq G. \quad (3.2.3.29)$$

The normalizer  $N_G(F_i)$  determines the subgroups conjugate to  $F_i$  under  $G$  (see Example 3.2.3.10). The number  $m$  of subgroups conjugate to a subgroup  $F_i$  under  $G$  equals the index of  $N_G(F_i)$  in  $G$ :

$$m = [G : N_G(F_i)] = |G| : |N_G(F_i)|, \quad (3.2.3.30)$$

where the last equation holds for finite  $G$  and  $F_i$ .

Normalizers of the subgroups of crystallographic point groups are available in Table 3.4.2.7 and in the software *GI★KoBo-1* under *Subgroups\View\Twinning Group*.

3.2.3.2.6. Normal subgroups

Among subgroups of a group, a special role is played by normal subgroups. A subgroup  $H$  of  $G$  is a *normal (invariant, self-conjugate) subgroup* of  $G$  if and only if it fulfils any of the following conditions:

(1) The subgroup  $H$  of  $G$  has no conjugate subgroups under  $G$ . (No subscript is therefore needed in the symbol of a normal subgroup  $H$ .)

(2) The normalizer  $N_G(H)$  of  $H$  equals the group  $G$ ,

$$N_G(H) = G. \quad (3.2.3.31)$$

(3) Every element  $g$  of  $G$  commutes with  $H$ , or, equivalently, each left coset  $gH$  equals the right coset  $Hg$ :

$$gH = Hg \text{ for every } g \in G. \quad (3.2.3.32)$$

For a normal subgroup  $H$  of a group  $G$  a special symbol  $\triangleleft$  is often used instead of  $\subset$ ,  $H \triangleleft G$ .