3.2.3.2.7. *Halving subgroups and dichromatic (black-and-white) groups*

Any subgroup H of a group G of index 2, called a *halving subgroup*, is a normal subgroup. The decomposition of G into left cosets of H consists of two left cosets,

$$G = H \cup gH. \qquad (3.2.3.33)$$

Sometimes it is convenient to distinguish elements of the coset gH from elements of the halving subgroup H. This can be achieved by attaching a sign (usually written as a superscript) to all elements of the coset. We shall use for this purpose the sign $\spadesuit$. To aid understanding, we shall also mark for a while the elements of the group H with another sign, $\heartsuit$. The multiplication law for these 'decorated elements' can be written in the following form:

$$g_1^{\heartsuit} g_2^{\heartsuit} = g_3^{\heartsuit}, \quad g_4^{\heartsuit} g_5^{\spadesuit} = g_6^{\spadesuit}, \quad g_7^{\spadesuit} g_8^{\heartsuit} = g_9^{\spadesuit}, \quad g_{10}^{\spadesuit} g_{11}^{\spadesuit} = g_{12}^{\heartsuit}. \qquad (3.2.3.34)$$

Now we replace the label $\heartsuit$ by a dummy 'no mark' sign (*i.e.* we remove $\heartsuit$), but we still keep in mind the multiplication rules (3.2.3.34). Then the decomposition (3.2.3.33) becomes

$$G = H \cup g^{\spadesuit} H, \qquad (3.2.3.33a)$$

since the coset $g^{\spadesuit}H$ assembles all marked elements and H consists of all bare elements of the group G.

The sign $\spadesuit$ can carry useful additional information, *e.g.* the application of labelled operations $g^{\spadesuit}$ is connected with some changes or new effects, whereas the application of a bare operation brings about no such changes or effects.

The label $\spadesuit$ can be replaced by various signs which can have different meanings. Thus in Chapter 3.3 a prime $'$ signifies a nontrivial twinning operation, in Chapter 1.5 it is associated with time inversion in magnetic structures, and in black-and-white patterns or structures a prime denotes an operation which exchanges black and white 'colours' (the qualifier 'black-and-white' concerns group operations, but not the black-and-white pattern itself). In Chapter 3.4, a star * denotes a transposing operation which exchanges two domain states, while underlining signifies an operation exchanging two sides of an interface and underlined operations with a star signify twinning operations of a domain twin. Various interpretations of the label attached to the symbol of an operation have given rise to several designations of groups with partition (3.2.3.34): *black-and-white*, *dichromatic*, *magnetic*, *anti-symmetry*, *Shubnikov* or *Heech–Shubnikov* and other *groups*. For more details see Opechowski (1986).

3.2.3.2.8. *Double cosets*

Let F_1 and H_1 be two proper subgroups of the group G. The set of all distinct products hg_jf, where g_j is a fixed element of the group G and f and h run over all elements of the subgroups F_1 and H_1, respectively, is called a *double coset of F_1 and H_1 in G*. The symbol of this double coset is $H_1g_jF_1$,

$$F_1 g_j H_1 = \{fg_jh | \forall f \in F_1, \forall h \in H_1\},$$
$$g_j \in G,\ F_1 \subset G,\ H_1 \subset G, \qquad (3.2.3.35)$$

where the sign $\forall$ means 'for all'.

In the symmetry analysis of domain structures, only double cosets with $H_1 = F_1$ are used. We shall, therefore, formulate subsequent definitions and statements only for this special type of double coset.

The fixed element g_j is called the *representative of the double coset $F_1g_jF_1$*. Any element of a double coset can be chosen as its representative.

Two double cosets are either identical or disjoint.

Proposition 3.2.3.6. The union of all distinct double cosets constitutes a partition of G and is called the *decomposition of the group G into double cosets of F_1*, since $F_1F_1 = F_1$. If the set of double cosets of F_1 in G is finite, then the decomposition of G into the double cosets of F_1 can be written as

$$G = F_1g_1F_1 \cup F_1g_2F_1 \cup \ldots \cup F_1g_qF_1. \qquad (3.2.3.36)$$

For the representative g_1 of the first double coset $F_1g_jF_1$ the unit element e is usually chosen, $g_1 = e$. Then the first double coset is identical with the subgroup F_1.

A double coset $F_1g_jF_1$ consists of left cosets of the form fg_jF_1, where $f \in F_1$. The number r of left cosets of F_1 in the double coset $F_1g_jF_1$ is (Hall, 1959)

$$r = [F_1 : F_{1j}], \qquad (3.2.3.37)$$

where

$$F_{1j} = F_1 \cap g_jF_1g_j^{-1}. \qquad (3.2.3.38)$$

The following definitions and statements are used in Chapter 3.4 for the double cosets $F_1g_jF_1$ [for derivations and more details, see Janovec (1972)].

The inverse $(F_1g_jF_1)^{-1}$ of a double coset $F_1g_jF_1$ is a double coset $F_1g_j^{-1}F_1$, which is either identical or disjoint with the double coset $F_1g_jF_1$. The double coset that is its own inverse is called an *invertible* (*self-inverse*, *ambivalent*) *double coset*. The double coset that is disjoint with its inverse is called a *non-invertible* (*polar*) *double coset* and the double cosets $F_1g_jF_1$ and $(F_1g_jF_1)^{-1} = F_1g_j^{-1}F_1$ are called *complementary polar double cosets*.

The inverse left coset $(g_jF_1)^{-1}$ contains representatives of all left cosets of the double coset $F_1g_j^{-1}F_1$. If a left coset g_jF_1 belongs to an invertible double coset, then $(g_jF_1)^{-1}$ contains representatives of left cosets constituting the double coset $F_1g_jF_1$. If a left coset g_jF_1 belongs to a non-invertible double coset, then $(g_jF_1)^{-1}$ contains representatives of left cosets constituting the complementary double coset $(F_1g_jF_1)^{-1}$.

A double coset consisting of only one left coset,

$$F_1g_jF_1 = g_jF_1, \qquad (3.2.3.39)$$

is called a simple *double coset*. A double coset $F_1g_jF_1$ is *simple* if and only if the inverse $(g_jF_1)^{-1}$ of the left coset g_jF_1 is again a left coset. For an invertible simple double coset $g_jF_1 = (g_jF_1)^{-1}$.

The union of all simple double cosets $F_1g_jF_1 = g_jF_1$ in the double coset decomposition of G (3.2.3.36) constitutes the normalizer $N_G(F_1)$ (Speiser, 1927).

A double coset that comprises more than one left coset will be called a *multiple double coset*. Four types of double cosets FgF are displayed in Table 3.2.3.1. The double coset decompositions of all crystallographic point groups are available in the software *GI★KoBo*-1 under *Subgroups\View\Twinning Group*.

Double cosets and the decomposition (3.2.3.36) of a group in double cosets are mathematical tools for partitioning a set of pairs of objects into equivalent classes (see Section 3.2.3.3.6). Such a division enables one to find possible twin laws and different types of domain walls that can appear in a domain structure resulting from a phase transition with a given symmetry descent (see Chapters 3.3 and 3.4).

More detailed introductions to group theory can be found in Budden (1972), Janssen (1973), Ledermann (1973), Rosen (1995), Shubnikov & Koptsik (1974), Vainshtein (1994) and Vainshtein *et*

Table 3.2.3.1. *Four types of double cosets*

	$FgF = gF$	$FgF \neq gF$
$FgF = (FgF)^{-1}$	Invertible simple	Invertible multiple
$FgF \cap (FgF)^{-1} = \emptyset$	Non-invertible simple	Non-invertible multiple