symmetry $G$ can be identified with all different orbits of subgroups of $G$.

In a similar manner, points of the order-parameter space and tensor-parameter space from Examples [sC] 3.2.3.16 and [oT] 3.2.3.21 can be divided into strata which are characterized by the orbits of possible stabilizers.

Next, we formulate three propositions that are essential in the symmetry analysis of domain structures presented in Section 3.4.2.

#### 3.2.3.3.4. *Orbits and left cosets*

*Proposition 3.2.3.23*. Let $G$ be a finite group, $\mathsf{A}$ a $G$-set and $I_G(\mathbf{S}_1) \equiv F_1$ the stabilizer of an object $\mathbf{S}_1$ of the set $\mathsf{A}$, $\mathbf{S}_1 \in \mathsf{A}$. The objects of the orbit

$$G\mathbf{S}_1 = \{\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_j, \ldots, \mathbf{S}_n\} \quad (3.2.3.67)$$

and the left cosets $g_jF_1$ of the decomposition of $G$,

$$G = g_1F_1 \cup g_2F_1 \cup \ldots \cup g_jF_1 \cup \ldots \cup g_nF_1 = \bigcup_{j=1}^{n} g_jF_1, \quad (3.2.3.68)$$

are in a one-to-one correspondence,

$$\mathbf{S}_j \leftrightarrow g_jF_1, \quad F_1 = I_G(\mathbf{S}_1),\ j = 1, 2, \ldots, n. \quad (3.2.3.69)$$

(See *e.g.* Kerber, 1991, 1999; Kopský, 1983; Lang, 1965.) The derivation of the bijection (3.2.3.69) consists of two parts:

(i) All operations of a left coset $g_jF_1$ transform $\mathbf{S}_1$ into the same $\mathbf{S}_j = g_j\mathbf{S}_1$, since $g_j\mathbf{S}_1 = g_j(F_1\mathbf{S}_1) = (g_jF_1)\mathbf{S}_1$, where we use the relation

$$\begin{aligned} F_1\mathbf{S}_1 &= \{f_1, f_2, \ldots, f_q\}\mathbf{S}_1 \\ &= \{f_1\mathbf{S}_1, f_2\mathbf{S}_1, \ldots, f_q\mathbf{S}_1\} \\ &= \{\mathbf{S}_1, \mathbf{S}_1, \ldots, \mathbf{S}_1\} = \{\mathbf{S}_1\} = \mathbf{S}_1, \end{aligned} \quad (3.2.3.70)$$

which in the second line contains a generalization of the group action and in the third line reflects Definition 3.2.3.1 of a set as a collection of distinguishable objects, $\{\mathbf{S}_1, \mathbf{S}_1, \ldots, \mathbf{S}_1\} = \mathbf{S}_1 \cup \mathbf{S}_1 \ldots \cup \mathbf{S}_1 = \mathbf{S}_1$.

(ii) Any $g_r \in G$ that transforms $\mathbf{S}_1$ into $\mathbf{S}_j = g_j\mathbf{S}_1$ belongs to the left coset $g_jF_1$, since from $g_j\mathbf{S}_1 = g_r\mathbf{S}_1$ it follows that $g_r^{-1}g_j\mathbf{S}_1 = \mathbf{S}_1$, *i.e.* $g_r^{-1}g_j \in F_1$, which, according to the left coset criterion, holds if and only if $g_r$ and $g_j$ belong to the same left coset $g_jF_1$.

We note that the orbit $G\mathbf{S}_1$ depends on the stabilizer $I_G(\mathbf{S}_1) = F_1$ of the object $\mathbf{S}_1$ and not on the 'eigensymmetry' of $\mathbf{S}_1$.

From Proposition 3.2.3.23 follow two corollaries:

*Corollary 3.2.3.24*. The order $n$ of the orbit $G\mathbf{S}_1$ equals the index of the stabilizer $I_G(\mathbf{S}_1) = F_1$ in $G$,

$$n = [G : I_G(\mathbf{S}_1)] = [G : F_1] = |G| : |F_1|, \quad (3.2.3.71)$$

where the last part of the equation applies to point groups only.

*Corollary 3.2.3.25*. All objects of the orbit $G\mathbf{S}_1$ can be generated by successive application of representatives of all left cosets $g_jF_1$ in the decomposition of $G$ [see (3.2.3.68)] to the object $\mathbf{S}_1$, $\mathbf{S}_j = g_j\mathbf{S}_1$, $j = 1, 2, \ldots, n$. The orbit $G\mathbf{S}_1$ can therefore be expressed explicitly as

$$G\mathbf{S}_1 = \{\mathbf{S}_1, g_2\mathbf{S}_1, \ldots, g_j\mathbf{S}_1, \ldots, g_n\mathbf{S}_1\}, \quad (3.2.3.72)$$

where the operations $g_1 = e, g_2, \ldots, g_j, \ldots, g_n$ (left transversal to $F_1$ in $G$) are the representatives of left cosets in the decomposition (3.2.3.68).

*Example [oP] 3.2.3.26*. The number of equivalent points of the point form $GX$ ($G$ orbit of the point $X$) is called a *multiplicity* $m_G(X)$ of this point,

$$m_G(X) = |G| : |I_G(X)|. \quad (3.2.3.73)$$

The multiplicity of a point of general position equals the order $|G|$ of the group $G$, since in this case $I_G(X) = e$, a trivial group. Then points of the orbit $GX$ and the operations of $G$ are in a one-to-one correspondence. The multiplicity of a point of special position is smaller than the order $|G|$, $m_G(X) < |G|$, and the operations of $G$ and the points of the orbit $GX$ are in a many-to-one correspondence. Points of a stratum have the same multiplicity; one can, therefore, talk about the multiplicity of the Wyckoff position [see *IT* A (2002)]. If $G$ is a space group, the point orbit has to be confined to the volume of the primitive unit cell (Wondratschek, 1995).

*Example [oC] 3.2.3.27*. Corollaries 3.2.3.24 and 3.2.3.25 applied to domain states represent the basic relations of domain-structure analysis. According to (3.2.3.71), the index $n$ of the stabilizer $I_G(\mathbf{S}_1)$ in the parent group $G$ gives the number of domain states in the orbit $G\mathbf{S}_1$ and the relations (3.2.3.72) and (3.2.3.68) give a recipe for constructing domain states of this orbit.

*Example [oT] 3.2.3.28*. If $\mu^{(1)}$ is a principal tensor parameter associated with the symmetry descent $G \supset F_1$, then there is a one-to-one correspondence between the elements of the orbit of single domain states $G\mathbf{S}_1 = \{\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_j, \ldots, \mathbf{S}_n\}$ and the elements of the orbit of the principal order parameter (points) $G\mu^{(1)} = \{\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(j)}, \ldots, \mu^{(n)}\}$ (see Example [oT] 3.2.3.21),

$$\mathbf{S}_j \leftrightarrow g_jF_1 \leftrightarrow \mu^{(j)}, \quad j = 1, 2, \ldots, n. \quad (3.2.3.74)$$

Therefore, single domain states of the orbit $G\mathbf{S}_1$ can be represented by the principal tensor parameter of the orbit $G\mu^{(1)}$.

*Example [oS] 3.2.3.29*. Consider a subgroup $F_1$ of a group $G$. Since the stabilizer of $F_1$ in $G$ is the normalizer $N_G(F_1)$ (see Example [sS] 3.2.3.17), the number $m$ of conjugate subgroups is, according to (3.2.3.71),

$$m = [G : N_G(F_1)] = |G| : |N_G(F_1)|, \quad (3.2.3.75)$$

where the last part of the equation applies to point groups only. The orbit of conjugate subgroups is

$$GF_1 = \{F_1, h_2F_1h_2^{-1}, \ldots, h_jF_1h_j^{-1}, \ldots h_mF_1h_m^{-1}\}, \quad j = 1, 2, \ldots, m, \quad (3.2.3.76)$$

where the operations $h_1 = e, h_2, \ldots, h_j, \ldots, h_m$ are the representatives of left cosets in the decomposition

$$G = N_G(F_1) \cup h_2N_G(F_1) \cup \ldots \cup h_jN_G(F_1) \cup \ldots \cup h_mN_G(F_1). \quad (3.2.3.77)$$

#### 3.2.3.3.5. *Intermediate subgroups and partitions of an orbit into suborbits*

*Proposition 3.2.3.30*. Let $G\mathbf{S}_1$ be a $G$ orbit from Proposition 3.2.3.23 and $L_1$ an intermediate group,

$$F_1 \subset L_1 \subset G. \quad (3.2.3.78)$$

A successive decomposition of $G$ into left cosets of $L_1$ and $L_1$ into left cosets of $F_1$ [see (3.2.3.25)] introduces a two-indices rela-