*International Tables for Crystallography* (2013). Vol. D, Section 1.1.4.10.6, pp. 27–29.



Furthermore, the left-hand term of (1.1.4.11) remains unchanged if we interchange the indices $i$ and $j$. The terms on the right-hand side therefore also remain unchanged, whatever the value of $T_{ll}$ or $T_{kl}$. It follows that

$$s_{ijll} = s_{jill}$$
$$s_{ijkl} = s_{ijlk} = s_{jikl} = s_{jilk}.$$

Similar relations hold for $c_{ijkl}$, $Q_{ijkl}$, $p_{ijkl}$ and $\pi_{ijkl}$: the submatrices **2** and **3**, **4** and **7**, **5**, **6**, **8** and **9**, respectively, are equal.

Equation (1.4.1.11) can be rewritten, introducing the coefficients of the Voigt strain matrix:

$$S_\alpha = S_{ii} = \sum_l s_{iill} T_{ll} + \sum_{k\neq l} (s_{iikl} + s_{iilk}) T_{kl} \quad (\alpha = 1, 2, 3)$$
$$S_\alpha = S_{ij} + S_{ji} = \sum_l (s_{ijll} + s_{jill}) T_{ll} + \sum_{k\neq l} (s_{ijkl} + s_{ijlk} + s_{jikl} + s_{jilk}) T_{kl} \quad (\alpha = 4, 5, 6).$$

We shall now introduce a two-index notation for the elastic compliances, according to the following conventions:

$$\left.\begin{array}{lll} i = j; & k = l; & s_{\alpha\beta} = s_{iill} \\ i = j; & k \neq l; & s_{\alpha\beta} = s_{iikl} + s_{iilk} \\ i \neq j; & k = l; & s_{\alpha\beta} = s_{ijkk} + s_{jikk} \\ i \neq j; & k \neq l; & s_{\alpha\beta} = s_{ijkl} + s_{jikl} + s_{ijlk} + s_{jilk}. \end{array}\right\} \quad (1.1.4.12)$$

We have thus associated with the fourth-rank tensor a square $6 \times 6$ matrix with 36 coefficients:

| $\alpha$ \ $\beta$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 11 | 12 | 13 | 14 | 15 | 16 |
| 2 | 21 | 22 | 23 | 24 | 25 | 26 |
| 3 | 31 | 32 | 33 | 34 | 35 | 36 |
| 4 | 41 | 42 | 43 | 44 | 45 | 46 |
| 5 | 51 | 52 | 53 | 54 | 55 | 56 |
| 6 | 61 | 62 | 63 | 64 | 65 | 66 |

One can translate relation (1.1.4.12) using the $9 \times 9$ matrix representing $s_{ijkl}$ by adding term by term the coefficients of submatrices **2** and **3**, **4** and **7** and **5**, **6**, **8** and **9**, respectively:

$$\begin{pmatrix} \mathbf{1} \\ \hline \mathbf{2+3} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \hline \mathbf{4+7} & \mathbf{5+6+8+9} \end{pmatrix} \times \begin{pmatrix} \mathbf{1} \\ \hline \mathbf{2+3} \end{pmatrix}$$

Using the two-index notation, equation (1.1.4.9) becomes

$$S_\alpha = s_{\alpha\beta} T_\beta. \quad (1.1.4.13)$$

A similar development can be applied to the other fourth-rank tensors $\pi_{ijkl}$, which will be replaced by $6 \times 6$ matrices with 36 coefficients, according to the following rules.

(i) *Elastic stiffnesses*, $c_{ijkl}$ and *elasto-optic coefficients*, $p_{ijkl}$:

$$\begin{pmatrix} \mathbf{1} \\ \hline \mathbf{2} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \hline \mathbf{4} & \mathbf{5} \end{pmatrix} \times \begin{pmatrix} \mathbf{1} \\ \hline \mathbf{2} \end{pmatrix}$$

where

$$c_{\alpha\beta} = c_{ijkl}$$
$$p_{\alpha\beta} = p_{ijkl}.$$

(ii) *Piezo-optic coefficients*, $\pi_{ijkl}$:

$$\begin{pmatrix} \mathbf{1} \\ \hline \mathbf{2} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{2+3} \\ \hline \mathbf{4} & \mathbf{5+6} \end{pmatrix} \times \begin{pmatrix} \mathbf{1} \\ \hline \mathbf{2} \end{pmatrix}$$

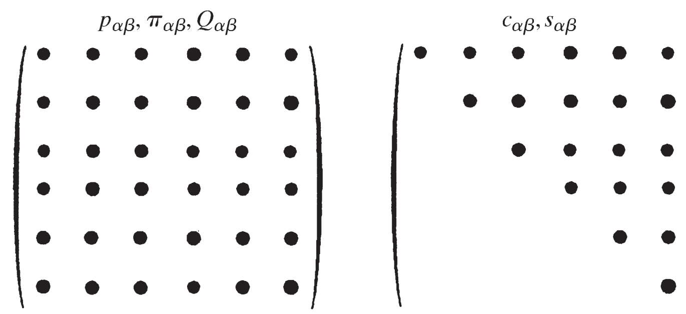
where

$$\left.\begin{array}{lll} i = j; & k = l; & \pi_{\alpha\beta} = \pi_{iill} \\ i = j; & k \neq l; & \pi_{\alpha\beta} = \pi_{iikl} + \pi_{iilk} \\ i \neq j; & k = l; & \pi_{\alpha\beta} = \pi_{ijkk} = \pi_{jikk} \\ i \neq j; & k \neq l; & \pi_{\alpha\beta} = \pi_{ijkl} + \pi_{jikl} = \pi_{ijlk} + \pi_{jilk}. \end{array}\right\}$$

(iii) *Electrostriction coefficients*, $Q_{ijkl}$: same relation as for the elastic compliances.

### 1.1.4.10.6. *Independent components of the matrix associated with a fourth-rank tensor according to the following point groups*

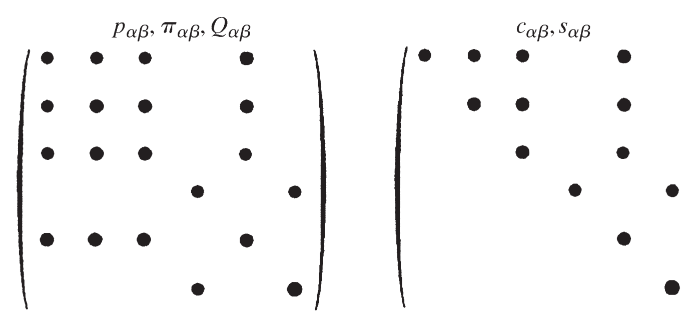
#### 1.1.4.10.6.1. *Triclinic system, groups* $\bar{1}$, 1




36 independent components — 21 independent components

#### 1.1.4.10.6.2. *Monoclinic system*

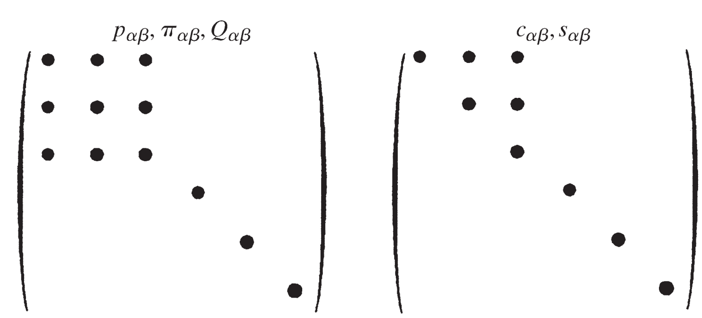
Groups $2/m$, 2, $m$, twofold axis parallel to $Ox_2$:




20 independent components — 13 independent components

#### 1.1.4.10.6.3. *Orthorhombic system*

Groups $mmm$, $2mm$, 222:

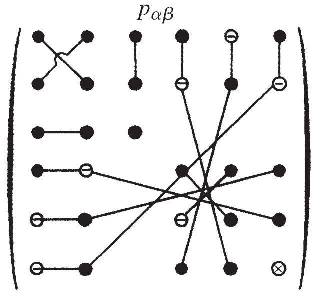



12 independent components — 9 independent components

1.1.4.10.6.4. *Trigonal system*

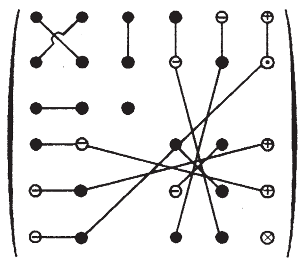
(i) Groups 3, $\bar{3}$:

$p_{\alpha\beta}$



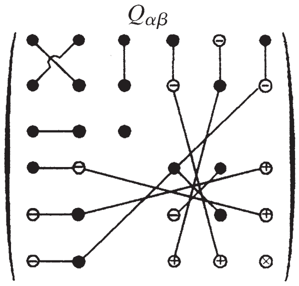
12 independent components

$\pi_{\alpha\beta}$



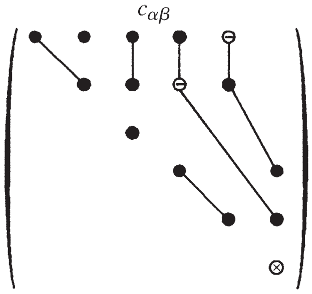
12 independent components

$Q_{\alpha\beta}$



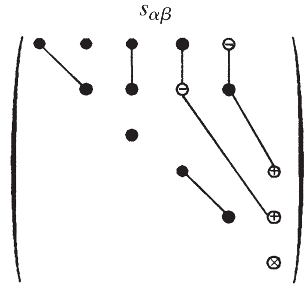
12 independent components

$c_{\alpha\beta}$



7 independent components

$s_{\alpha\beta}$

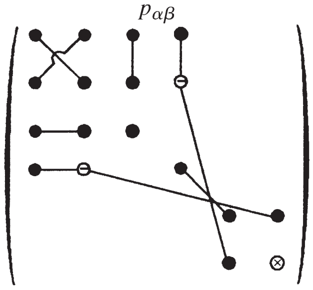


7 independent components

where ⊖ is a component numerically equal but opposite in sign to the heavy dot component to which it is linked; ⊕ is a component equal to twice the heavy dot component to which it is linked; ⊙ is a component equal to minus twice the heavy dot component to which it is linked; ⊗ is equal to $1/2(p_{11} - p_{12})$, $(\pi_{11} - \pi_{12})$, $2(Q_{11} - Q_{12})$, $1/2(c_{11} - c_{12})$ and $2(s_{11} - s_{12})$, respectively.

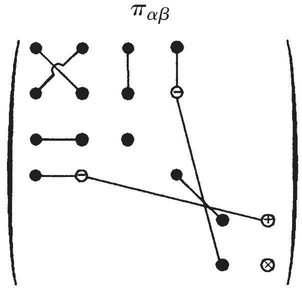
(ii) Groups 32, 3*m*, $\bar{3}m$:

$p_{\alpha\beta}$



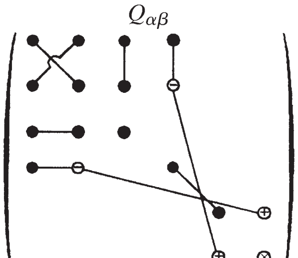
8 independent components

$\pi_{\alpha\beta}$



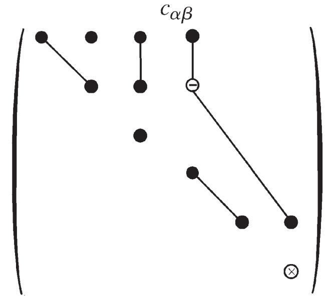
8 independent components

$Q_{\alpha\beta}$



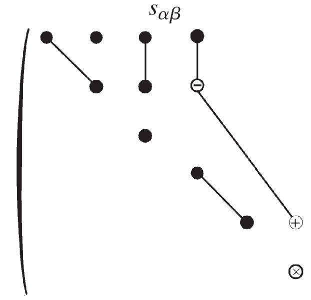
8 independent components

$c_{\alpha\beta}$



6 independent components

$s_{\alpha\beta}$



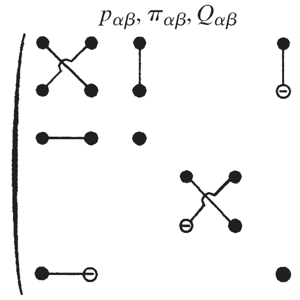
6 independent components

with the same conventions. The sign of $c_{14}$ depends on the orientation of the $Ox_1$ axis.

1.1.4.10.6.5. *Tetragonal system*

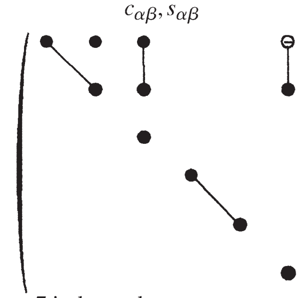
(i) Groups 4, $\bar{4}$ and 4/*m*:

$p_{\alpha\beta}, \pi_{\alpha\beta}, Q_{\alpha\beta}$



10 independent components

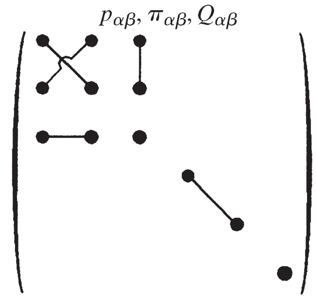
$c_{\alpha\beta}, s_{\alpha\beta}$



7 independent components

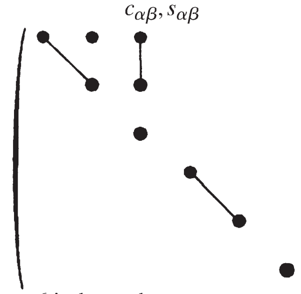
(ii) Groups 422, 4*mm*, $\bar{4}2m$ and 4/*mm*:

$p_{\alpha\beta}, \pi_{\alpha\beta}, Q_{\alpha\beta}$



7 independent components

$c_{\alpha\beta}, s_{\alpha\beta}$

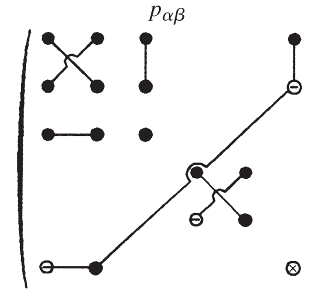


6 independent components

1.1.4.10.6.6. *Hexagonal system*

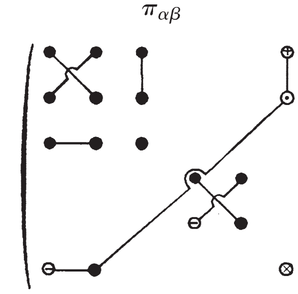
(i) Groups 6, $\bar{6}$ and 6/*m*:

$p_{\alpha\beta}$



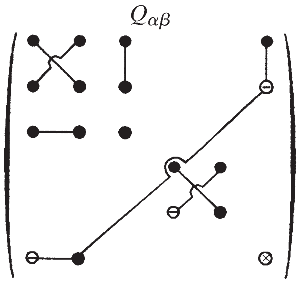
8 independent components

$\pi_{\alpha\beta}$



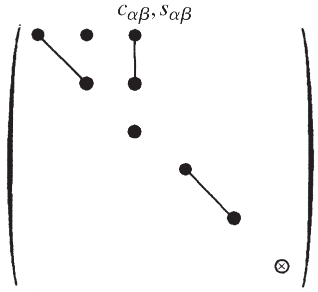
8 independent components

$Q_{\alpha\beta}$



8 independent components

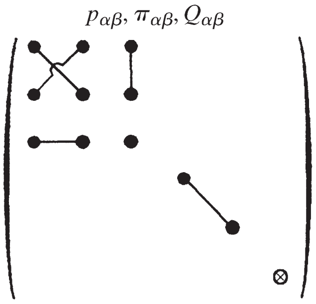
$c_{\alpha\beta}, s_{\alpha\beta}$



5 independent components

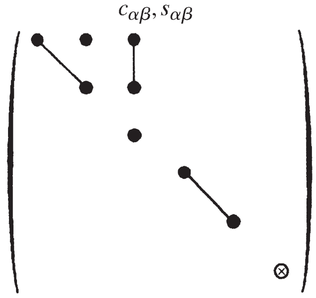
(ii) Groups 622, 6*mm*, $\bar{6}2m$ and 6/*mm*:

$p_{\alpha\beta}, \pi_{\alpha\beta}, Q_{\alpha\beta}$



6 independent components

$c_{\alpha\beta}, s_{\alpha\beta}$

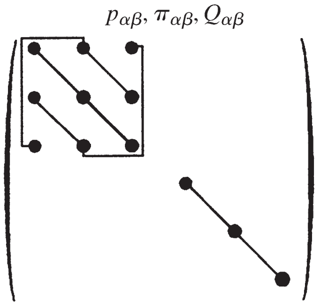


5 independent components

1.1.4.10.6.7. *Cubic system*

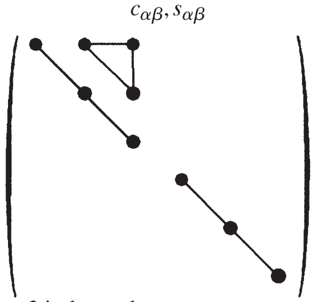
(i) Groups 23 and 3*m*:

$p_{\alpha\beta}, \pi_{\alpha\beta}, Q_{\alpha\beta}$



4 independent components

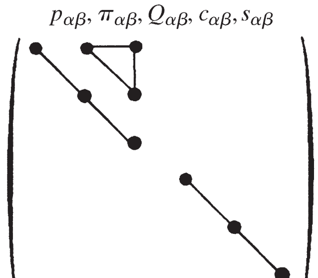
$c_{\alpha\beta}, s_{\alpha\beta}$



3 independent components

(ii) Groups 432, $\bar{4}3m$ and $m\bar{3}m$:

$p_{\alpha\beta}, \pi_{\alpha\beta}, Q_{\alpha\beta}, c_{\alpha\beta}, s_{\alpha\beta}$

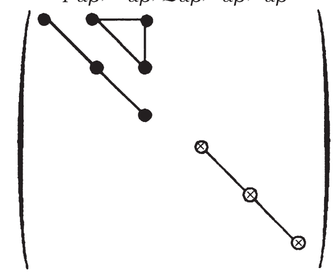


3 independent components

1.1.4.10.6.8. *Spherical system*

For all tensors

$p_{\alpha\beta}, \pi_{\alpha\beta}, Q_{\alpha\beta}, c_{\alpha\beta}, s_{\alpha\beta}$



2 independent components

1.1.4.10.7. *Reduction of the number of independent components of axial tensors of rank 2*

It was shown in Section 1.1.4.5.3.2 that axial tensors of rank 2 are actually tensors of rank 3 antisymmetric with respect to two indices. The matrix of independent components of a tensor such that

$$g_{ijk} = -g_{jik}$$

is given by

$$\left(\begin{array}{ccc|ccc|ccc} & 122 & 133 & 123 & 131 & & 132 & & 121 \\ -121 & & 223 & & 231 & -122 & 232 & -123 & \\ -131 & -232 & & -233 & & -132 & & -133 & -231 \end{array}\right).$$

The second-rank axial tensor $g_{kl}$ associated with this tensor is defined by

$$g_{kl} = \tfrac{1}{2}\varepsilon_{ijk} g_{ijl}.$$

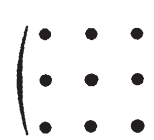
For instance, the piezomagnetic coefficients that give the magnetic moment $M_i$ due to an applied stress $T_\alpha$ are the components of a second-rank axial tensor, $\Lambda_{i\alpha}$ (see Section 1.5.7.1):

$$M_i = \Lambda_{i\alpha} T_\alpha.$$

1.1.4.10.7.1. *Independent components according to the following point groups*

(i) *Triclinic system*

(*a*) Group 1:

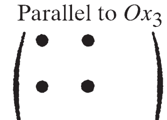


(*b*) Group $\bar{1}$: all components are equal to zero.

(ii) *Monoclinic system*

(*a*) Group 2:

Parallel to $Ox_3$



Parallel to $Ox_2$