International Tables for Crystallography (2013). Vol. D, Section 1.2.7.4, pp. 67–70.

However, many entries for the three-dimensional point groups are simply integers.

The program provides the following information as rows above the characters of the irreducible representation:

(1) Representative elements of the conjugacy classes expressed in terms of the generators $a, b, \ldots$.

(2) The number of elements of each class.

(3) The order of the elements of the classes: the lowest positive power of an element that equals the identity.

Below the character table, the following information is displayed:

(1) In the mth row after the square character table, the class to which the $(m+1)$th powers of the elements from this column belong is given. If a conjugacy class has elements of order p, then only the $p-1$ first entries are given, because in the column there exists p periodicity.

(2) The determinant of the three-dimensional matrix for the element of the point group (or the elements of the conjugacy class). This is the character of an irreducible representation.

(3) Finally, the character of the vector representation is given.

As an example, the generalized character table for the three-dimensional point group $4mm$ is given in Table 1.2.7.1.

The data connected with a character table can be seen by choosing 'view character table'. The characters of the irreducible representations, the determinant representation and the vector representation are shown in the main window after selection of 'accept character table'. From the character of these representations, characters of other representations may be calculated. The results are added as rows to the table, which is shown after each calculation.

Calculations using rows from the table may have one or more arguments. Operations with one argument will produce, for example, the decomposition into irreducible components, the character of the pth power, the symmetrized or antisymmetrized square, or the character of the corresponding physical (real) representation. Operations with two or more arguments yield products and sums of characters. The arguments of a unitary, binary or multiple operation are selected by clicking on the button in front of the corresponding characters. If the result is a new character (*e.g.* the product of two characters), it is added as a row to the list of characters. If the result is not a character (*e.g.* the decomposition into irreducible components), the result is given on the worksheet.

Suppose one wants to determine the number of elastic constants for a material with cubic 432 symmetry. After selecting the character table for the group 432, one clicks on the button in front of 'vector representation' in the character table. This yields the character of the three-dimensional vector representation of the group. The character of the symmetrized square is obtained by selecting 'symmetrized square'. This gives the character of a six-dimensional representation. Determining the number of times the trivial representation occurs by selecting 'decompose' gives the number of free parameters in the metric tensor, *i.e.* 1. Clicking on 'symmetrized square' for the character of the six-dimensional representation gives the character of a 21-dimensional representation. Decomposition yields the multiplicity 3 for the trivial representation, which means that there are three independent tensor elements for a tensor of symmetry type ((0 1)(2 3)), which in turn means that there are three elastic constants for the group 432 (see Table 1.2.6.9). For the explicit determination of the independent tensor elements, the tensor module of the program should be used.

Table 1.2.7.1. *Data connected with the character table for point group* $4mm$

e	a	a^2	b	ab
1	2	1	2	2
1	4	2	2	2
1	1	1	1	1
1	1	1	−1	−1
1	−1	1	1	−1
1	−1	1	−1	1
2	0	−2	0	0
1	3	1	1	1
	2			
	1			
1	1	1	−1	−1
3	1	−1	1	1

Of course, many kinds of calculations unrelated to tensors can be carried out using the character module. Examples include the calculation of selection rules in spectroscopy or the splitting of energy levels under a symmetry-breaking perturbation.

1.2.7.4. *Algorithms*

1.2.7.4.1. *Construction of a basis*

As a basis for a tensor space without permutation symmetry, one may choose one consisting of non-commutative monomials. It has d^r elements, where d is the dimension and r is the rank. In two dimensions, these are x, y for $r=1$, xx, xy, yx, yy for $r=2$ and $xxx, xxy, xyx, xyy, yxx, yxy, yyx, yyy$ for $r=3$. Note that $xy \neq yx$.

If there is permutation symmetry among the indices $i_1, \ldots, i_p$, only polynomials $x_{i_1}x_{i_2}\ldots x_r$ occur in the basis for which $i_1 \leq i_2 \leq \ldots \leq i_p$. Then $x_{i_1}x_{i_2} = x_{i_2}x_{i_1}$. If there is antisymmetry among these indices, one has the condition $i_1 < i_2 < \ldots < i_p$ and $x_{i_1}x_{i_2} = -x_{i_2}x_{i_1}$. Therefore, in two dimensions, the basis for tensors of type (1 3)2 is $xxx, xxy, xyx, xyy, yxy, yyy$ and for those of type [1 3]2 it is xxy, xyy. These bases can be obtained from the general basis by elimination.

1.2.7.4.2. *Action of the generators of the point group G on the basis*

The transformation of the monomial $x_ix_j\ldots$ under the matrix $g \in G$ is given by the polynomial

$$\left[\sum_{m=1}^{d} g_{im}x_m\right] \times \left[\sum_{n=1}^{d} g_{jn}x_n\right]\ldots,$$

which is in principle non-commutative. This polynomial can be written as a sum of the monomials in the basis taking into account the eventual (anti)symmetry of xy and yx. In this way, basis element (a monomial) e_i is transformed to

$$ge_i = \sum_{j=1}^{d} M(g)_{ji}e_j.$$

To each generator of G corresponds such an action matrix M.

The action matrix changes if one considers pseudotensors. In the case of pseudotensors, the previous equation changes to

$$ge_i = \mathrm{Det}(g)\sum_{j=1}^{d} M(g)_{ji}e_j.$$

The function $\mathrm{Det}(g)$ is just a one-dimensional representation of the group G. The determinant is either $+1$ or -1.

1.2.7.4.3. *Diagonalization of the action matrix and determination of the invariant tensor*

An invariant element of the tensor space under the group G is a vector v that is left invariant under each generator:

$$\begin{pmatrix} M_1 - E \\ M_2 - E \\ \vdots \\ M_s - E \end{pmatrix} v = \Omega v = 0.$$

If the number of generators is one, $\Omega = M - E$. This equation is solved by diagonalization:

$$P\Omega QQ^{-1}v = DQ^{-1}v = 0,$$

where $D_{ij} = d_i\delta_{ij}$. The dimension of the solution space is the number of elements d_i that are equal to zero. The corresponding rows of Q form a basis for the solution space. (See example further on.)

1.2.7.4.4. *Determination of the vector representation*

For a point group G, its isomorphism class and its character table are known. For each conjugacy class, a representative element is given as word $A_1A_2\ldots$ where the A_i's correspond to generators. Replacing the letters by the generating matrices, one obtains as product a matrix for which the trace is the character of the vector representation in the conjugacy class. The characters of all conjugacy classes being known, the representation can be decomposed into irreducible components by means of

$$m_\alpha = (1/|G|)\sum_i n_i\chi_\alpha^*(i)\chi(i),$$

where α labels the irreducible representations (the row number in the character table), m_α the number of times the representation α occurs, $|G|$ the order of the group G, n_i the number of elements in the ith conjugacy class (given as the second row in the character table), $\chi_\alpha(i)$ the cyclotomic in the ith row and αth column of the character table, and $\chi(i)$ the calculated character in the ith conjugacy class.

1.2.7.4.5. *Determination of tensor products and their decomposition*

Given a character (for an irreducible representation from the character table, or for the vector representation, for example), the character of the standard rank n tensor is the nth power of the character and can be decomposed with the multiplicity formula for m_α given above.

Fully symmetrized or antisymmetrized tensor products have characters given by

$$\begin{aligned}
n=2:\ \chi^\pm(R) &= \frac{1}{2!}\big(\chi(R)^2 \pm \chi(R^2)\big)\\
n=3:\ \chi^\pm(R) &= \frac{1}{3!}\big(\chi(R)^3 \pm 3\chi(R^2)\chi(R) + 2\chi(R^3)\big)\\
n=4:\ \chi^\pm(R) &= \frac{1}{4!}\big(\chi(R)^4 \pm 6\chi(R^2)\chi(R)^2 + 3\chi(R^2)^2\\
&\quad + 8\chi(R^3)\chi(R) \pm 6\chi(R^4)\big)\\
n=5:\ \chi^\pm(R) &= \frac{1}{5!}\big(\chi(R)^5 \pm 10\chi(R^2)\chi(R)^3 + 15\chi(R^2)^2\chi(R)\\
&\quad + 20\chi(R^3)\chi(R)^2 \pm 20\chi(R^3)\chi(R^2)\\
&\quad \pm 30\chi(R^4)\chi(R) + 24\chi(R^5)\big)\\
n=6:\ \chi^\pm(R) &= \frac{1}{6!}\big(\chi(R)^6 \pm 15\chi(R^2)\chi(R)^4 + 45\chi(R^2)^2\chi(R)^2\\
&\quad + 40\chi(R^3)^2 \pm 15\chi(R^2)^3 + 40\chi(R^3)\chi(R)^3\\
&\quad \pm 120\chi(R^3)\chi(R^2)\chi(R) \pm 90\chi(R^4)\chi(R)^2\\
&\quad + 90\chi(R^4)\chi(R^2) + 144\chi(R^5)\chi(R)\\
&\quad \pm 120\chi(R^6)\big).
\end{aligned}$$

From this follows immediately the dimension of the subspaces of symmetric and antisymmetric tensors:

$$\begin{aligned}
n=2 &: \frac{1}{2}(d^2 \pm d)\\
n=3 &: \frac{1}{6}(d^3 \pm 3d^2 + 2d)\\
n=4 &: \frac{1}{24}(d^4 \pm 6d^3 + 11d^2 \pm 6d)\\
n=5 &: \frac{1}{120}(d^5 \pm 10d^4 + 35d^3 \pm 50d^2 + 24d)\\
n=6 &: \frac{1}{720}(d^6 \pm 15d^5 + 85d^4 \pm 225d^3 + 274d^2 \pm 120d).
\end{aligned}$$

The general expression for arbitrary rank can be determined as follows. (See also Section 1.2.2.7)

(1) If n is the rank, the first step is to determine all possible decompositions

$$n = \sum_{i=1}^n f_i$$

with non-negative integers f_i satisfying $f_i \le f_{i-1}$.

(2) For each such decomposition $m = 1, \ldots n_{\rm tot}$ there is a term

$$P_m = \prod_{i=1}^p \binom{N_i}{f_i}(f_i - 1)!,$$

where $N_1 = n$, $N_i = N_{i-1} - f_{i-1}$ $(i > 1)$ and p is the number of nonzero integers f_i.

(3) If there are equal values of f_i in the mth decomposition, P_m should be divided by $t!$ for each t-tuple of equal values $(f_{k+1} = \ldots = f_{k+t})$.

(4) The sign of the term P_m is $+1$ for a symmetrized power and

$$\prod_{i=1}^p (-1)^{(f_i-1)}$$

for an antisymmetrized power.

(5) The expression for the character of the (anti)symmetrized power then is

$$\chi^\pm(R) = (1/M!)\sum_{m=1}^{n_{\rm tot}} {\rm sign}_m P_m \prod_{i=1}^p \chi(R^{f_i}).$$

1.2.7.4.6. *Invariant tensors*

Once one has the character of the properly symmetrized tensor, the number of invariants is just m_1, the number of times the trivial representation occurs in the decomposition.

Example (1). Dimension 3, rank 3, symmetry type (123), group 3. Basis: *xxx*, *xxy*, *xxz*, *xyy*, *xyz*, *xzz*, *yyy*, *yyz*, *yzz*, *zzz*. Under

$$\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0 \end{pmatrix}$$

the basis elements go to *yyy*, *yyz*, *yyx*, *yzz*, *yzx*, *yxx*, *zzz*, *zzx*, *zxx*, *xxx*, respectively, and these are equivalent to *yyy*, *yyz*, *xyy*, *yzz*, *xyz*, *xxy*, *zzz*, *xzz*, *xxz*, *xxx*, respectively. This gives the ten-dimensional matrix

$$M = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0 \end{pmatrix}.$$

Then $P(M-E)Q = D$, with D diagonal. There are four diagonal elements of D which are zero, and the invariant tensors correspond to the corresponding four columns of the matrix Q. The invariant polynomials are

$$xxx + yyy + zzz, \quad xxy + xzz + yyz, \quad xxz + yzz + xyy, \quad xyz.$$

Example (2). Dimension 2, rank 2, symmetry type (12). Group generated by

$$\begin{pmatrix} 0 & -1\\ 1 & -1 \end{pmatrix}.$$

Basis xx, xy, yy goes to yy, $-xy+yy$, $xx-2xy+yy$. This gives

$$M = \begin{pmatrix} 0 & 0 & 1\\ 0 & -1 & -2\\ 1 & 1 & 1 \end{pmatrix}.$$

Because

$$\begin{pmatrix} 1&0&0\\ 2&1&2\\ 1&0&1 \end{pmatrix}(M-E)\begin{pmatrix} 1&-1&1\\ 0&1&0\\ 0&-1&1 \end{pmatrix} = \begin{pmatrix} -1&0&0\\ 0&0&0\\ 0&0&2 \end{pmatrix},$$

the invariant tensor corresponds to the second column of Q, which as a polynomial reads $-xx+xy-yy$. This can be written with the tensor T_{ij} as

$$-xx+xy-yy = -\sum_{i,j} T_{ij}x_ix_j, \quad T_{ij} = \begin{pmatrix} 1 & -\frac{1}{2}\\ -\frac{1}{2} & 1 \end{pmatrix}.$$

This tensor T is invariant under the group.

Example (3). Dimension 3, rank 2, tensor type (12). Group generated by matrix([[0 −1 0][1 0 0][0 0 1]]). The basis xx, xy, xz, yy, yz, zz goes under the generator to yy, $-xy$, $-yz$, xx, xz, zz. The solution of $(M-E)v=0$ is

$$\alpha_1(xx+yy) + \alpha_2 zz.$$

The matrix D has two zeros on the diagonal.

Example (4). Dimension 3, rank 3, type (123). Same group as in Example (3). Basis xxx, xxy, xxz, xyy, xyz, xzz, yyy, yyz, yzz, zzz. The solution

$$\alpha_1(xxz+yyz) + \alpha_2 zzz$$

corresponds to a tensor with relations $T_{113} = T_{223}$, $T_{111} = T_{112} = T_{122} = T_{123} = T_{133} = T_{222} = T_{233} = 0$.

Example (5). Dimension 3, rank 4, type ((12)(34)). Not only $i_1 \le i_2$ and $i_3 \le i_4$, but also $(i_1 i_2)$, should come lexicographically before $(i_3 i_4)$. Basis $xxxx$, $xxxy$, $xxxz$, $xxyy$, $xxyz$, $xxzz$, $xyxy$, $xyxz$, $xyyy$, $xyyz$, $xyzz$, $xzxz$, $xzyy$, $xzyz$, $xzzz$, $yyyy$, $yyyz$, $yyzz$, $yzyz$, $yzzz$, $zzzz$. Under the same group as in example (3), there are seven invariants. Invariant polynomial:

$$\alpha_1(xxxx+yyyy) + \alpha_2(xxxy-xyyy) + \alpha_3 xxyy + \alpha_4 xyxy + \alpha_5 zzzz + \alpha_6(xxzz+yyzz) + \alpha_7(xzxz+yzyz).$$

Table 1.2.7.2. *Calculation with characters*

Generator	Composite character	Characters				Decomposition
$\begin{pmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{pmatrix}$	R	E	A	AA		
	$\chi(R)$	3	0	0		
	$\chi(R)^3$	27	0	0		
	$\chi(R^2)$	3	0	0		
	$\chi(R^2)\chi(R)$	9	0	0		
	$\chi(R^3)$	3	3	3		
Example (1)	$\frac{1}{6}(\chi(R)^3 + 3\chi(R^2)\chi(R) + 2\chi(R^3))$	10	1	1		$4D_1 + 3D_2 + 3D_3$
$\begin{pmatrix} 0&-1\\ 1&-1 \end{pmatrix}$	R	E	A	AA		
	$\chi(R)$	2	−1	−1		
	$\chi(R)^2$	4	1	1		
	$\chi(R^2)$	2	−1	−1		
Example (2)	$\frac{1}{2}(\chi(R)^2 + \chi(R^2))$	3	0	0		$D_1 + D_2 + D_3$
$\begin{pmatrix} 0&-1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix}$	R	E	A	AA	AAA	
	$\chi(R)$	3	1	−1	1	
	$\chi(R)^2$	9	1	1	1	
	$\chi(R^2)$	3	−1	3	−1	
Example (3)	$\frac{1}{2}(\chi(R)^2 + \chi(R^2))$	6	0	2	0	$2D_1 + D_2 + 2D_3 + D_4$
As above	$\chi(R)$	3	1	−1	1	
Example (4)	$\chi(R)^3$	27	1	−1	1	
	$\chi(R^2)$	3	−1	3	−1	
	$\chi(R^2)\chi(R)$	9	−1	−3	−1	
	$\chi(R^3)$	3	1	−1	1	
	$\frac{1}{6}(\chi(R)^3 + 3\chi(R^2)\chi(R) + 2\chi(R^3))$	10	0	−2	0	$2D_1 + 3D_2 + 2D_3 + 3D_4$
As above	$\chi(R)$	3	1	−1	1	
Example (5)	$\frac{1}{2}(\chi(R)^2 + \chi(R^2)) = \chi_s(R)$	6	0	2	0	
	$\chi_s(R)^2$	36	0	4	0	
	$\chi_s(R^2)$	6	2	6	2	
	((12)(34))	21	1	5	1	$7D_1 + 4D_2 + 6D_3 + 4D_4$
As above, example (6)	$\frac{1}{2}(\chi(R)^2 - \chi(R^2))$	3	1	−1	1	$D_1 + D_2 + D_4$
As above, example (7)	$\frac{1}{6}(\chi(R)^3 - 3\chi(R^2)\chi(R) + 2\chi(R^3))$	1	1	1	1	D_1

This corresponds to the tensor relations

$$\begin{array}{lll} \mathrm{T}_{xxxx} = -\mathrm{T}_{yyyy} & \mathrm{T}_{xxxy} = \mathrm{T}_{xyyy} & \mathrm{T}_{xxxz} = 0 \\ \mathrm{T}_{xxyz} = 0 & \mathrm{T}_{xxzz} = \mathrm{T}_{yyzz} & \mathrm{T}_{xyxz} = 0 \\ \mathrm{T}_{xyyz} = 0 & \mathrm{T}_{xyzz} = 0 & \mathrm{T}_{xzxz} = \mathrm{T}_{yzyz} \\ \mathrm{T}_{xzyy} = 0 & \mathrm{T}_{xzyz} = 0 & \mathrm{T}_{xzzz} = 0 \\ \mathrm{T}_{yyyz} = 0 & \mathrm{T}_{yzzz} = 0 & \end{array}$$

$$\rightarrow \begin{pmatrix} \alpha_1 & \alpha_3 & \alpha_6 & 0 & 0 & \alpha_2 \\ \alpha_3 & \alpha_1 & \alpha_6 & 0 & 0 & -\alpha_2 \\ \alpha_6 & \alpha_6 & \alpha_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_7 & 0 \\ \alpha_2 & -\alpha_2 & 0 & 0 & 0 & \alpha_4 \end{pmatrix}.$$

The latter form is that of an elastic tensor with the usual convention $1 = xx$, $2 = yy$, $3 = zz$, $4 = yz$, $5 = xz$, $6 = xy$.

Example (6). Dimension 3, rank 2, type [12]. The same group as in example (3). Basis $xy, xz, yz \rightarrow -yx, -yz, xz$, which are equivalent to xy, $-yz$, xz. The transformation in the tensor space is

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix} v = 0:$$

$$v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \sim xy.$$

There is just one invariant antisymmetric polynomial $xy = -yx$ corresponding to the tensor

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Example (7). Dimension 3, rank 3, type [123]. Basis xyz invariant under the group: $xyz \rightarrow -yxz \sim xyz$. The corresponding tensor is the fully antisymmetric rank 3 tensor: $\mathrm{T}_{ijk} = 1$ if ijk is an even permutation of 123, $= -1$ if ijk is an odd permutation, and $= 0$ if two or three indices are equal (permutation tensor, see Section 1.1.3.7.2).

Example (8). Calculation with characters. See Table 1.2.7.2.

Example (9). The action matrix for a pseudotensor.

Take the group $4/m$ with generators

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Consider the rank 3 pseudotensor (123). The action matrix is determined from the action of the generators A and B on the basis:

	A	B
xxx	$-yyy$	$-xxx$
xxy	xyy	$-xxy$
xxz	yyz	xxz
xyy	$-xxy$	$-xyy$
xyz	$-xyz$	xyz
xzz	$-yzz$	$-xzz$
yyy	xxx	$-yyy$
yyz	xxz	yyz
yzz	xzz	$-yzz$
zzz	zzz	zzz

Therefore, the action matrix becomes

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

After diagonalization, one finds two nonzero elements on the diagonal:

$$zzz = a; \quad xxz = yyz = b;$$
$$xxx = xxy = xyy = xyz = xzz = yyy = yzz = 0.$$

1.2.8. Glossary

$\mathrm{T}_{i_1 \ldots i_n}$	tensor of rank n
$O(n)$	orthogonal group
$\mathbb{Z}$	ring of integers
$\mathbf{e}_i$	basis vectors
g	metric tensor
K	point group
R	orthogonal transformation
C_m	cyclic group of order m
$SO(n)$	special orthogonal group
$\mathbb{Z}^+$	positive integers
D_n	dihedral group of order n
E	unit transformation, matrix or element
I	inversion
$D(K)$	representation of K
$\Gamma(K)$	matrix representation of K
$\lvert K \rvert$	order of K
$\oplus$	sum of spaces or operators
$\otimes$	tensor product
$\in$	element of
$\mathbf{a}_i$	basis of space or lattice

references