International Tables for Crystallography (2013). Vol. D, Section 1.2.8, pp. 70–71.

This corresponds to the tensor relations

$$\begin{array}{lll} \mathrm{T}_{xxxx} = -\mathrm{T}_{yyyy} & \mathrm{T}_{xxxy} = \mathrm{T}_{xyyy} & \mathrm{T}_{xxxz} = 0 \\ \mathrm{T}_{xxyz} = 0 & \mathrm{T}_{xxzz} = \mathrm{T}_{yyzz} & \mathrm{T}_{xyxz} = 0 \\ \mathrm{T}_{xyyz} = 0 & \mathrm{T}_{xyzz} = 0 & \mathrm{T}_{xzxz} = \mathrm{T}_{yzyz} \\ \mathrm{T}_{xzyy} = 0 & \mathrm{T}_{xzyz} = 0 & \mathrm{T}_{xzzz} = 0 \\ \mathrm{T}_{yyyz} = 0 & \mathrm{T}_{yzzz} = 0 & \end{array}$$

$$\rightarrow \begin{pmatrix} \alpha_1 & \alpha_3 & \alpha_6 & 0 & 0 & \alpha_2 \\ \alpha_3 & \alpha_1 & \alpha_6 & 0 & 0 & -\alpha_2 \\ \alpha_6 & \alpha_6 & \alpha_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_7 & 0 \\ \alpha_2 & -\alpha_2 & 0 & 0 & 0 & \alpha_4 \end{pmatrix}.$$

The latter form is that of an elastic tensor with the usual convention $1 = xx$, $2 = yy$, $3 = zz$, $4 = yz$, $5 = xz$, $6 = xy$.

Example (6). Dimension 3, rank 2, type [12]. The same group as in example (3). Basis $xy, xz, yz \rightarrow -yx, -yz, xz$, which are equivalent to $xy, -yz, xz$. The transformation in the tensor space is

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix} v = 0:$$

$$v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \sim xy.$$

There is just one invariant antisymmetric polynomial $xy = -yx$ corresponding to the tensor

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Example (7). Dimension 3, rank 3, type [123]. Basis xyz invariant under the group: $xyz \rightarrow -yxz \sim xyz$. The corresponding tensor is the fully antisymmetric rank 3 tensor: $\mathrm{T}_{ijk} = 1$ if ijk is an even permutation of 123, $= -1$ if ijk is an odd permutation, and $= 0$ if two or three indices are equal (permutation tensor, see Section 1.1.3.7.2).

Example (8). Calculation with characters. See Table 1.2.7.2.

Example (9). The action matrix for a pseudotensor.

Take the group $4/m$ with generators

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Consider the rank 3 pseudotensor (123). The action matrix is determined from the action of the generators A and B on the basis:

	A	B
xxx	$-yyy$	$-xxx$
xxy	xyy	$-xxy$
xxz	yyz	xxz
xyy	$-xxy$	$-xyy$
xyz	$-xyz$	xyz
xzz	$-yzz$	$-xzz$
yyy	xxx	$-yyy$
yyz	xxz	yyz
yzz	xzz	$-yzz$
zzz	zzz	zzz

Therefore, the action matrix becomes

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

After diagonalization, one finds two nonzero elements on the diagonal:

$$zzz = a; \quad xxz = yyz = b;$$

$$xxx = xxy = xyy = xyz = xzz = yyy = yzz = 0.$$

1.2.8. Glossary

$\mathrm{T}_{i_1 \ldots i_n}$	tensor of rank n
$O(n)$	orthogonal group
$\mathbb{Z}$	ring of integers
$\mathbf{e}_i$	basis vectors
g	metric tensor
K	point group
R	orthogonal transformation
C_m	cyclic group of order m
$SO(n)$	special orthogonal group
$\mathbb{Z}^+$	positive integers
D_n	dihedral group of order n
E	unit transformation, matrix or element
I	inversion
$D(K)$	representation of K
$\Gamma(K)$	matrix representation of K
$\lvert K \rvert$	order of K
$\oplus$	sum of spaces or operators
$\otimes$	tensor product
$\in$	element of
$\mathbf{a}_i$	basis of space or lattice

V^*	dual space
S	basis transformation
χ	character
$\chi(R)$	value of χ at R
C_i	conjugacy class
χ_α	irreducible character
m_α	multiplicity
N	order of K
d_α	dimension of irreducible representation α
n_i	order of class C_i
c_{ijk}	class multiplication constants
T	tetrahedral group
O	octahedral group
I	icosahedral group
$P(K)$	projective representation
$W_i(A_1, \ldots, A_p)$	word in generators A_j
K^d	double group
$E(n)$	Euclidean group
$g = \{R\|\mathbf{a}\}$	element of $E(n)$
$T(n)$	translation group in n dimensions
Λ	lattice
Λ^*	reciprocal lattice
$\mathbf{a}(R)$	translation vector system
$\mathbf{k}$	vector in dual space
$G_\mathbf{k}$	group of $\mathbf{k}$
$K_\mathbf{k}$	point group of $G_\mathbf{k}$
ω	factor system
$\mathrm{Det}(R)$	determinant of R
$\begin{pmatrix} \alpha & \beta & \vert & \gamma & \\ i & j & \vert & k & \ell \end{pmatrix}$	Clebsch–Gordan coefficients
θ	time-reversal operator

References

Altmann, S. L. & Herzig, P. (1994). *Point-Group Theory Tables*. Oxford: Clarendon Press.

Butler, P. H. (1981). *Point Group Symmetry Applications*. New York & London: Plenum Press.

International Tables for Crystallography (2005). Vol. A. *Space-Group Symmetry*, edited by Th. Hahn. Heidelberg: Springer.

Janssen, T. (1973). *Crystallographic Groups*. Amsterdam: North-Holland.

Kovalev, O. V. (1987). *Representations of the Crystallographic Space Groups*. New York: Gordon and Breach.

Miller, S. C. & Love, W. F. (1967). *Tables of Irreducible Representations of Space Groups and Co-representations of Magnetic Space Groups*. Boulder, Colorado: Pruett Press.

Stokes, H. T. & Hatch, D. M. (1988). *Isotropy Subgroups of the 230 Crystallographic Space Groups*. Singapore: World Scientific.