$$M = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0 \end{pmatrix}.$$

Then $P(M-E)Q = D$, with D diagonal. There are four diagonal elements of D which are zero, and the invariant tensors correspond to the corresponding four columns of the matrix Q. The invariant polynomials are

$$xxx + yyy + zzz, \quad xxy + xzz + yyz, \quad xxz + yzz + xyy, \quad xyz.$$

Example (2). Dimension 2, rank 2, symmetry type (12). Group generated by

$$\begin{pmatrix} 0 & -1\\ 1 & -1 \end{pmatrix}.$$

Basis xx, xy, yy goes to yy, $-xy+yy$, $xx-2xy+yy$. This gives

$$M = \begin{pmatrix} 0 & 0 & 1\\ 0 & -1 & -2\\ 1 & 1 & 1 \end{pmatrix}.$$

Because

$$\begin{pmatrix} 1&0&0\\ 2&1&2\\ 1&0&1 \end{pmatrix}(M-E)\begin{pmatrix} 1&-1&1\\ 0&1&0\\ 0&-1&1 \end{pmatrix} = \begin{pmatrix} -1&0&0\\ 0&0&0\\ 0&0&2 \end{pmatrix},$$

the invariant tensor corresponds to the second column of Q, which as a polynomial reads $-xx+xy-yy$. This can be written with the tensor T_{ij} as

$$-xx+xy-yy = -\sum_{i,j} T_{ij}x_ix_j, \quad T_{ij} = \begin{pmatrix} 1 & -\frac{1}{2}\\ -\frac{1}{2} & 1 \end{pmatrix}.$$

This tensor T is invariant under the group.

Example (3). Dimension 3, rank 2, tensor type (12). Group generated by matrix([[0 −1 0][1 0 0][0 0 1]]). The basis xx, xy, xz, yy, yz, zz goes under the generator to yy, $-xy$, $-yz$, xx, xz, zz. The solution of $(M-E)v = 0$ is

$$\alpha_1(xx+yy) + \alpha_2 zz.$$

The matrix D has two zeros on the diagonal.

Example (4). Dimension 3, rank 3, type (123). Same group as in Example (3). Basis xxx, xxy, xxz, xyy, xyz, xzz, yyy, yyz, yzz, zzz. The solution

$$\alpha_1(xxz+yyz) + \alpha_2 zzz$$

corresponds to a tensor with relations $\mathrm{T}_{113} = \mathrm{T}_{223}$, $\mathrm{T}_{111} = \mathrm{T}_{112} = \mathrm{T}_{122} = \mathrm{T}_{123} = \mathrm{T}_{133} = \mathrm{T}_{222} = \mathrm{T}_{233} = 0$.

Example (5). Dimension 3, rank 4, type ((12)(34)). Not only $i_1 \le i_2$ and $i_3 \le i_4$, but also $(i_1 i_2)$, should come lexicographically before $(i_3 i_4)$. Basis $xxxx$, $xxxy$, $xxxz$, $xxyy$, $xxyz$, $xxzz$, $xyxy$, $xyxz$, $xyyy$, $xyyz$, $xyzz$, $xzxz$, $xzyy$, $xzyz$, $xzzz$, $yyyy$, $yyyz$, $yyzz$, $yzyz$, $yzzz$, $zzzz$. Under the same group as in example (3), there are seven invariants. Invariant polynomial:

$$\alpha_1(xxxx+yyyy) + \alpha_2(xxxy-xyyy) + \alpha_3 xxyy + \alpha_4 xyxy + \alpha_5 zzzz + \alpha_6(xxzz+yyzz) + \alpha_7(xzxz+yzyz).$$

Table 1.2.7.2. *Calculation with characters*

Generator	Composite character	Characters				Decomposition
$\begin{pmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{pmatrix}$	R	E	A	AA		
	$\chi(R)$	3	0	0		
	$\chi(R)^3$	27	0	0		
	$\chi(R^2)$	3	0	0		
	$\chi(R^2)\chi(R)$	9	0	0		
	$\chi(R^3)$	3	3	3		
Example (1)	$\frac{1}{6}(\chi(R)^3 + 3\chi(R^2)\chi(R) + 2\chi(R^3))$	10	1	1		$4D_1 + 3D_2 + 3D_3$
$\begin{pmatrix} 0&-1\\ 1&-1 \end{pmatrix}$	R	E	A	AA		
	$\chi(R)$	2	−1	−1		
	$\chi(R)^2$	4	1	1		
	$\chi(R^2)$	2	−1	−1		
Example (2)	$\frac{1}{2}(\chi(R)^2 + \chi(R^2))$	3	0	0		$D_1 + D_2 + D_3$
$\begin{pmatrix} 0&-1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix}$	R	E	A	AA	AAA	
	$\chi(R)$	3	1	−1	1	
	$\chi(R)^2$	9	1	1	1	
	$\chi(R^2)$	3	−1	3	−1	
Example (3)	$\frac{1}{2}(\chi(R)^2 + \chi(R^2))$	6	0	2	0	$2D_1 + D_2 + 2D_3 + D_4$
As above	$\chi(R)$	3	1	−1	1	
Example (4)	$\chi(R)^3$	27	1	−1	1	
	$\chi(R^2)$	3	−1	3	−1	
	$\chi(R^2)\chi(R)$	9	−1	−3	−1	
	$\chi(R^3)$	3	1	−1	1	
	$\frac{1}{6}(\chi(R)^3 + 3\chi(R^2)\chi(R) + 2\chi(R^3))$	10	0	−2	0	$2D_1 + 3D_2 + 2D_3 + 3D_4$
As above	$\chi(R)$	3	1	−1	1	
Example (5)	$\frac{1}{2}(\chi(R)^2 + \chi(R^2)) = \chi_s(R)$	6	0	2	0	
	$\chi_s(R)^2$	36	0	4	0	
	$\chi_s(R^2)$	6	2	6	2	
	((12)(34))	21	1	5	1	$7D_1 + 4D_2 + 6D_3 + 4D_4$
As above, example (6)	$\frac{1}{2}(\chi(R)^2 - \chi(R^2))$	3	1	−1	1	$D_1 + D_2 + D_4$
As above, example (7)	$\frac{1}{6}(\chi(R)^3 - 3\chi(R^2)\chi(R) + 2\chi(R^3))$	1	1	1	1	D_1

  references