

1.3. ELASTIC PROPERTIES

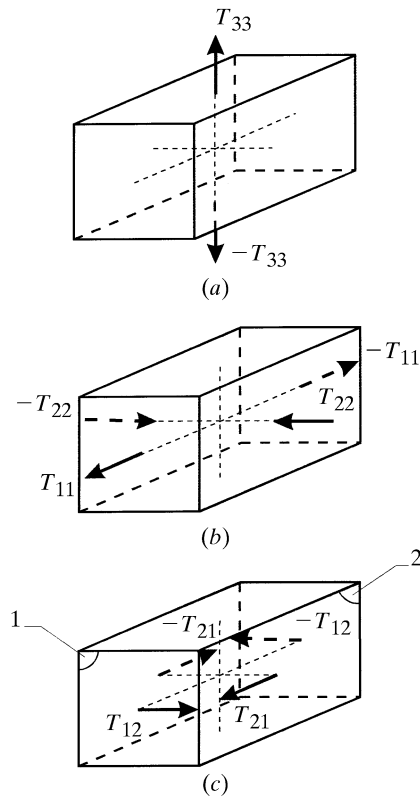


Fig. 1.3.2.5. Special forms of the stress tensor. (a) Uniaxial stress: the stress tensor has only one component,  $T_{33}$ ; (b) pure shear stress:  $T_{22} = -T_{11}$ ; (c) simple shear stress:  $T_{21} = T_{12}$ .

(apart from the volume forces mentioned earlier), the stress field inside the solid is such that at each point of the surface

$$T_{nj} = T_{ij}\alpha_i,$$

where the  $\alpha_j$ 's are the direction cosines of the normal to the surface at the point under consideration.

1.3.2.7. Local properties of the stress tensor

(i) *Normal stress and shearing stress*: let us consider a surface area element  $d\sigma$  within the solid, the normal  $\mathbf{n}$  to this element and the stress  $\mathbf{T}_n$  that is applied to it (Fig. 1.3.2.6).

The *normal stress*,  $\nu$ , is, by definition, the component of  $\mathbf{T}_n$  on  $\mathbf{n}$ ,

$$\nu = \mathbf{n}(\mathbf{T}_n \cdot \mathbf{n})$$

and the *shearing stress*,  $\tau$ , is the projection of  $\mathbf{T}_n$  on the surface area element,

$$\tau = \mathbf{n} \wedge (\mathbf{T}_n \wedge \mathbf{n}) = \mathbf{T}_n - \nu \mathbf{n}$$

(ii) *The stress quadric*: let us consider the bilinear form attached to the stress tensor:

$$f(\mathbf{y}) = T_{ij}y_i y_j.$$

The quadric represented by

$$f(\mathbf{y}) = \varepsilon$$

is called the stress quadric, where  $\varepsilon = \pm 1$ . It may be an ellipsoid or a hyperboloid. Referred to the principal axes, and using Voigt's notation, its equation is

$$y_i^2 T_i = \varepsilon.$$

To every direction  $\mathbf{n}$  of the medium, let us associate the radius vector  $\mathbf{y}$  of the quadric (Fig. 1.3.2.7) through the relation

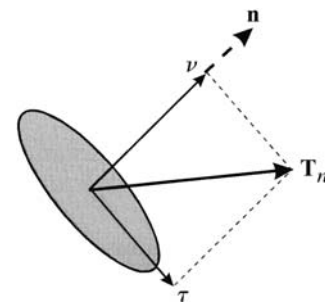


Fig. 1.3.2.6. Normal ( $\nu$ ) and shearing ( $\tau$ ) stress.

$$\mathbf{n} = k\mathbf{y}.$$

The stress applied to a small surface element  $d\sigma$  normal to  $\mathbf{n}$ ,  $\mathbf{T}_n$ , is

$$\mathbf{T}_n = k\nabla(f)$$

and the normal stress,  $\nu$ , is

$$\nu = \alpha_i T_i = 1/y^2,$$

where the  $\alpha_i$ 's are the direction cosines of  $\mathbf{n}$ .

(iii) *Principal normal stresses*: the stress tensor is symmetrical and has therefore real eigenvectors. If we represent the tensor with reference to a system of axes parallel to its eigenvectors, it is put in the form

$$\begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix}.$$

$T_1$ ,  $T_2$  and  $T_3$  are the principal normal stresses. The mean normal stress,  $T$ , is defined by the relation

$$T = (T_1 + T_2 + T_3)/3$$

and is an invariant of the stress tensor.

1.3.2.8. Energy density in a deformed medium

Consider a medium that is subjected to a stress field  $T_{ij}$ . It has sustained a deformation indicated by the deformation tensor  $S$ . During this deformation, the forces of contact have performed work and the medium has accumulated a certain elastic energy  $W$ . The knowledge of the energy density thus acquired is useful for studying the properties of the elastic constants. Let the medium deform from the deformation  $S_{ij}$  to the deformation  $S_{ij} + \delta S_{ij}$  under the influence of the stress field and let us evaluate the work of each component of the effort. Consider a small elementary rectangular parallelepiped of sides  $2\Delta x_1, 2\Delta x_2, 2\Delta x_3$  (Fig. 1.3.2.8). We shall limit our calculation to the components  $T_{11}$  and  $T_{12}$ , which are applied to the faces 1 and 1', respectively.

In the deformation  $\delta S$ , the point  $P$  goes to the point  $P'$ , defined by

$$\mathbf{PP}' = \mathbf{u}(\mathbf{r}).$$

A neighbouring point  $Q$  goes to  $Q'$  such that (Fig. 1.3.1.1)

$$\mathbf{PQ} = \Delta \mathbf{r}; \quad \mathbf{P}'Q' = \delta \mathbf{r}'.$$

The coordinates of  $\delta \mathbf{r}'$  are given by

$$\delta x'_i = \delta \Delta x_i + \delta S_{ij} \delta x_j.$$

Of sole importance is the relative displacement of  $Q$  with respect to  $P$  and the displacement that must be taken into account in calculating the forces applied at  $Q$ . The coordinates of the relative displacement are