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1.3. Elastic properties

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1.3.1. Strain tensor

1.3.1.1. *Introduction, the notion of strain field*

Let us consider a medium that undergoes a deformation. This means that the various points of the medium are displaced with respect to one another. Geometrical transformations of the medium that reduce to a translation of the medium as a whole will therefore not be considered. We may then suppose that there is an invariant point, O, whose position one can always return to by a suitable translation. A point P, with position vector $\mathbf{OP} = \mathbf{r}$, is displaced to the neighbouring point P' by the deformation defined by

$$\mathbf{PP}' = \mathbf{u}(\mathbf{r}).$$

The displacement vector $\mathbf{u}(\mathbf{r})$ constitutes a vector field. It is not a uniform field, unless the deformation reduces to a translation of the whole body, which is incompatible with the hypothesis that the medium undergoes a deformation. Let Q be a point that is near P before the deformation (Fig. 1.3.1.1). Then one can write

$$\mathrm{d}\mathbf{r} = \mathbf{PQ}; \quad \mathbf{r} + \mathrm{d}\mathbf{r} = \mathbf{OQ}.$$

After the deformation, Q is displaced to Q' defined by

$$\mathbf{QQ}' = \mathbf{u}(\mathbf{r} + \mathrm{d}\mathbf{r}).$$

In a deformation, it is more interesting in general to analyse the local, or relative, deformation than the absolute displacement. The relative displacement is given by comparing the vectors $\mathbf{P}'\mathbf{Q}' = \mathrm{d}\mathbf{r}'$ and $\mathbf{PQ}$. Thus, one has

$$\mathbf{P}'\mathbf{Q}' = \mathbf{P}'\mathbf{P} + \mathbf{PQ} + \mathbf{QQ}'.$$

Let us set

$$\left.\begin{aligned} \mathrm{d}\mathbf{r}' &= \mathrm{d}\mathbf{r} + \mathbf{u}(\mathbf{r} + \mathrm{d}\mathbf{r}) - \mathbf{u}(\mathbf{r}) \\ \mathrm{d}\mathbf{u} &= \mathbf{u}(\mathbf{r} + \mathrm{d}\mathbf{r}) - \mathbf{u}(\mathbf{r}) = \mathrm{d}\mathbf{r}' - \mathrm{d}\mathbf{r}. \end{aligned}\right\} \quad (1.3.1.1)$$

Replacing $\mathbf{u}(\mathbf{r} + \mathrm{d}\mathbf{r})$ by its expansion up to the first term gives

$$\left.\begin{aligned} \mathrm{d}u_i &= \frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j \\ \mathrm{d}x_i' &= \mathrm{d}x_i + \frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j. \end{aligned}\right\} \quad (1.3.1.2)$$

If we assume the Einstein convention (see Section 1.1.2.1), there is summation over j in (1.3.1.2) and (1.3.1.3). We shall further assume orthonormal coordinates throughout Chapter 1.3; variance is therefore not apparent and the positions of the indices have no meaning; the Einstein convention then only assumes repetition of a dummy index. The elements $\mathrm{d}x_i$ and $\mathrm{d}x_i'$ are the components of $\mathrm{d}\mathbf{r}$ and $\mathrm{d}\mathbf{r}'$, respectively. Let us put

$$M_{ij} = \partial u_i/\partial x_j; \;\; B_{ij} = M_{ij} + \delta_{ij},$$

where δ_{ij} represents the Kronecker symbol; the δ_{ij}'s are the components of matrix unity, I. The expressions (1.3.1.2) can also be written using matrices M and B:

$$\left.\begin{aligned} \mathrm{d}u_i &= M_{ij}\mathrm{d}x_j \\ \mathrm{d}x_i' &= B_{ij}\mathrm{d}x_j. \end{aligned}\right\} \quad (1.3.1.3)$$

The components of the tensor M_{ij} are nonzero, unless, as mentioned earlier, the deformation reduces to a simple translation. Two cases in particular are of interest and will be discussed in turn:

(i) The components M_{ij} are constants. In this case, the deformation is homogeneous.

(ii) The components M_{ij} are variables but are small compared with unity. This is the practical case to which we shall limit ourselves in considering an inhomogeneous deformation.

1.3.1.2. *Homogeneous deformation*

If the components M_{ij} are constants, equations (1.3.1.3) can be integrated directly. They become, to a translation,

$$\left.\begin{aligned} u_i &= M_{ij}x_j \\ x_i' &= B_{ij}x_j. \end{aligned}\right\} \quad (1.3.1.4)$$

1.3.1.2.1. *Fundamental property of the homogeneous deformation*

The fundamental property of the homogeneous deformation results from the fact that equations (1.3.1.4) are linear: a plane before the deformation remains a plane afterwards, a crystal lattice remains a lattice. Thermal expansion is a homogeneous deformation (see Chapter 1.4).

1.3.1.2.2. *Spontaneous strain*

Some crystals present a twin microstructure that is seen to change when the crystals are gently squeezed. At rest, the domains can have one of two different possible orientations and the influence of an applied stress is to switch them from one orientation to the other. If one measures the shape of the crystal lattice (the strain of the lattice) as a function of the applied stress, one obtains an elastic hysteresis loop analogous to the magnetic or electric hysteresis loops observed in ferromagnetic or ferroelectric crystals. For this reason, these materials are called *ferroelastic* (see Chapters 3.1 to 3.3 and Salje, 1990). The strain associated with one of the two possible shapes of the crystal when no stress is applied is called the macroscopic *spontaneous strain*.

1.3.1.2.3. *Cubic dilatation*

Let $\mathbf{e}_i$ be the basis vectors before deformation. On account of the deformation, they are transformed into the three vectors

$$\mathbf{e}_i' = B_{ij}\mathbf{e}_j.$$

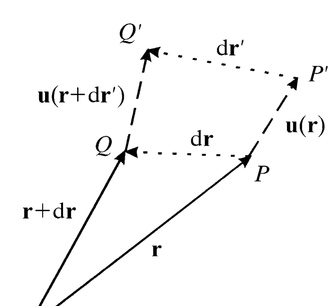

Fig. 1.3.1.1. Displacement vector, $\mathbf{u}(\mathbf{r})$.

The parallelepiped formed by these three vectors has a volume V' given by

$$V' = (\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = \Delta(B)(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \Delta(B)V,$$

where $\Delta(B)$ is the determinant associated with matrix B, V is the volume before deformation and

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot \mathbf{e}_3$$

represents a triple scalar product.

The relative variation of the volume is

$$\frac{V' - V}{V} = \Delta(B) - 1. \qquad (1.3.1.5)$$

It is what one calls the *cubic dilatation*. $\Delta(B)$ gives directly the volume of the parallelepiped that is formed from the three vectors obtained in the deformation when starting from vectors forming an orthonormal base.

1.3.1.2.4. *Expression of any homogeneous deformation as the product of a pure rotation and a pure deformation*

(i) *Pure rotation*: It is isometric. The moduli of the vectors remain unchanged and one direction remains invariant, the axis of rotation. The matrix B is unitary:

$$BB^T = 1.$$

(ii) *Pure deformation*: This is a deformation in which three orthogonal directions remain invariant. It can be shown that B is a symmetric matrix:

$$B = B^T.$$

The three invariant directions are those of the eigenvectors of the matrix; it is known in effect that the eigenvectors of a symmetric matrix are real.

(iii) *Arbitrary deformation*: the matrix B, representing an arbitrary deformation, can always be put into the form of the product of a unitary matrix B_1, representing a pure rotation, and a symmetric matrix B_2, representing a pure deformation. Let us put

$$B = B_1 B_2$$

and consider the transpose matrix of B:

$$B^T = B_2^T B_1^T = B_2 (B_1)^{-1}.$$

The product $B^T B$ is equal to

$$B^T B = (B_2)^2.$$

This shows that we can determine B_2 and therefore B_1 from B.

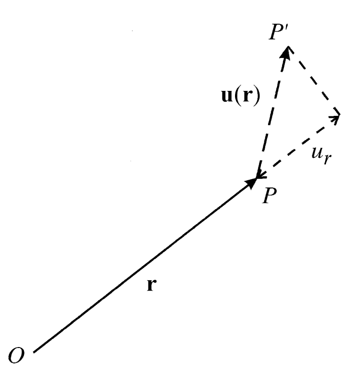

Fig. 1.3.1.2. Elongation, u_r/r.

1.3.1.2.5. *Quadric of elongations*

Let us project the displacement vector $\mathbf{u}(\mathbf{r})$ on the position vector **OP** (Fig. 1.3.1.2), and let u_r be this projection. The *elongation* is the quantity defined by

$$\frac{u_r}{r} = \frac{\mathbf{u} \cdot \mathbf{r}}{r^2} = \frac{M_{ij} x_i x_j}{r^2},$$

where x_1, x_2, x_3 are the components of $\mathbf{r}$. The elongation is the relative variation of the length of the vector $\mathbf{r}$ in the deformation. Let A and S be the antisymmetric and symmetric parts of M, respectively:

$$A = \frac{M - M^T}{2}; \quad S = \frac{M + M^T}{2}.$$

Only the symmetric part of M occurs in the expression of the elongation:

$$\frac{u_r}{r} = \frac{S_{ij} x_i x_j}{r^2}. \qquad (1.3.1.6)$$

The geometrical study of the elongation as a function of the direction of $\mathbf{r}$ is facilitated by introducing the quadric associated with M:

$$S_{ij} y_i y_j = \varepsilon, \qquad (1.3.1.7)$$

where ε is a constant. This quadric is called the *quadric of elongations*, Q. S is a symmetric matrix with three real orthogonal eigenvectors and three real eigenvalues, λ_1, λ_2, λ_3. If it is referred to these axes, equation (1.3.1.7) is reduced to

$$\lambda_1 (y_1)^2 \lambda_2 (y_2)^2 \lambda_3 (y_3)^2 = \varepsilon.$$

One can discuss the form of the quadric according to the sign of the eigenvalues λ_i:

(i) λ_1, λ_2, λ_3 have the same sign, and the sign of ε. The quadric is an ellipsoid (Fig. 1.3.1.3*a*). One chooses $\varepsilon = +1$ or $\varepsilon = -1$, depending on the sign of the eigenvalues.

(ii) λ_1, λ_2, λ_3 are of mixed signs: one of them is of opposite sign to the other two. One takes $\varepsilon = \pm 1$. The corresponding quadric is a hyperboloid whose asymptote is the cone

$$S_{ij} y_i y_j = 0.$$

According to the sign of ε, the hyperboloid will have one sheet outside the cone or two sheets inside the cone (Fig. 1.3.1.3*b*). If we wish to be able to consider any direction of the position vector $\mathbf{r}$ in space, it is necessary to take into account the two quadrics.

In order to follow the variations of the elongation u_r/r with the orientation of the position vector, one associates with $\mathbf{r}$ a vector $\mathbf{y}$, which is parallel to it and is defined by

$$\mathbf{y} = \mathbf{r}/k; \quad \mathbf{r} = k\mathbf{y},$$

where k is a constant. It can be seen that, in accordance with (1.3.1.6) and (1.3.1.7), the expression of the elongation in terms of y is

$$u_r/r = \varepsilon/y^2.$$

Thus, the elongation is inversely proportional to the square of the radius vector of the quadric of elongations parallel to **OP**. In practice, it is necessary to look for the intersection p of the parallel to **OP** drawn from the centre O of the quadric of elongations (Fig. 1.3.1.3*a*):

(i) The eigenvalues all have the same sign; the quadric Q is an ellipsoid: the elongation has the same sign in all directions in space, positive for $\varepsilon = +1$ and negative for $\varepsilon = -1$.

(ii) The eigenvalues have different signs; two quadrics are to be taken into account: the hyperboloids corresponding, respectively, to $\varepsilon = \pm 1$. The sign of the elongation is different according to whether the direction under consideration is outside or inside the asymptotic cone and intersects one or the other of the two hyperboloids.

Equally, one can connect the displacement vector $\mathbf{u}(\mathbf{r})$ directly with the quadric Q. Using the bilinear form

$$f(\mathbf{y}) = M_{ij} y_i y_j,$$

the gradient of $f(\mathbf{y})$, $\boldsymbol{\nabla}(f)$, has as components

$$\partial f / \partial y^i = M_{ij} y_j = u_i.$$

One recognizes the components of the displacement vector $\mathbf{u}$, which is therefore parallel to the normal to the quadric Q at the extremity of the radius vector **Op** parallel to $\mathbf{r}$.

The directions of the principal axes of Q correspond to the extremal values of y, *i.e.* to the stationary values (maximal or minimal) of the elongation. These values are the *principal elongations*.

If the deformation is a pure rotation

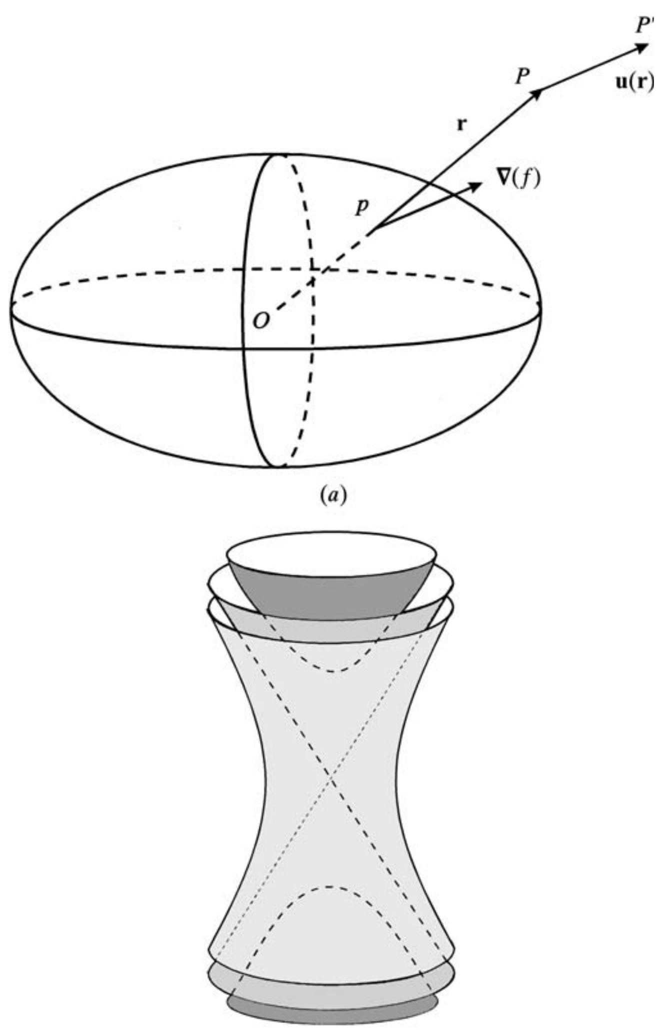

Fig. 1.3.1.3. Quadric of elongations. The displacement vector, $\mathbf{u}(\mathbf{r})$, at P in the deformed medium is parallel to the normal to the quadric at the intersection, p, of **OP** with the quadric. (*a*) The eigenvalues all have the same sign, the quadric is an ellipsoid. (*b*) The eigenvalues have mixed signs, the quadric is a hyperboloid with either one sheet (shaded in light grey) or two sheets (shaded in dark grey), depending on the sign of the constant ε [see equation (1.3.1.7)]; the cone asymptote is represented in medium grey. For a practical application, see Fig. 1.4.1.1.

$$B = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M = \begin{pmatrix} \cos\theta - 1 & \sin\theta & 0 \\ -\sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence we have

$$M_{ij} y_i y_j = (\cos\theta - 1)(y_1 - y_2) = \varepsilon.$$

The quadric Q is a cylinder of revolution having the axis of rotation as axis.

1.3.1.3. *Arbitrary but small deformations*

1.3.1.3.1. *Definition of the strain tensor*

If the deformation is small but arbitrary, *i.e.* if the products of two or more components of M_{ij} can be neglected with respect to unity, one can describe the deformation locally as a homogeneous asymptotic deformation. As was shown in Section 1.3.1.2.4, it can be put in the form of the product of a pure deformation corresponding to the symmetric part of M_{ij}, S_{ij}, and a pure rotation corresponding to the asymmetric part, A_{ij}:

$$\left.\begin{aligned} S_{ij} = S_{ji} &= \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\ A_{ij} = -A_{ji} &= \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \end{aligned}\right\} \qquad (1.3.1.8)$$

Matrix B can be written

$$B = I + A + S,$$

where I is the matrix identity. As the coefficients $\partial u_i / \partial x_j$ of M_{ij} are small, one can neglect the product $A \times S$ and one has

$$B = (I + A)(I + S).$$

$(I + S)$ is a symmetric matrix that represents a pure deformation. $(I + A)$ is an antisymmetric unitary matrix and, since A is small,

$$(I + A)^{-1} = (I - A).$$

Thus, $(I + A)$ represents a rotation. The axis of rotation is parallel to the vector with coordinates

$$\left.\begin{aligned} \Omega_1 &= \tfrac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) = A_{32} \\ \Omega_2 &= \tfrac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = A_{13} \\ \Omega_3 &= \tfrac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) = A_{21}, \end{aligned}\right\}$$

which is an eigenvector of $(I + A)$. The magnitude of the rotation is equal to the modulus of this vector.

In general, one is only interested in the pure deformation, *i.e.* in the form of the deformed object. Thus, one only wishes to know the quantities $(I + S)$ and the symmetric part of M. It is this symmetric part that is called the deformation tensor or the strain tensor. It is very convenient for applications to use the simplified notation due to Voigt:

$$S_1 = \frac{\partial u_1}{\partial x_1}; \qquad S_2 = \frac{\partial u_2}{\partial x_2}; \qquad S_3 = \frac{\partial u_3}{\partial x_3};$$

$$S_4 = \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}; \quad S_5 = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}; \quad S_6 = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}.$$

One may note that

$$S_1 = S_{11}; \quad S_2 = S_{22}; \quad S_3 = S_{33};$$
$$S_4 = S_{23} + S_{32}; \quad S_5 = S_{31} + S_{13}; \quad S_6 = S_{12} + S_{21}.$$

The Voigt strain matrix S is of the form

$$\begin{pmatrix} S_1 & S_6 & S_5 \\ S_6 & S_2 & S_4 \\ S_5 & S_4 & S_3 \end{pmatrix}.$$

1.3.1.3.2. *Geometrical interpretation of the coefficients of the strain tensor*

Let us consider an orthonormal system of axes with centre P. We remove nothing from the generality of the following by limiting ourselves to a planar problem and assuming that point P' to which P goes in the deformation lies in the plane x_1Px_2 (Fig. 1.3.1.4). Let us consider two neighbouring points, Q and R, lying on axes Px_1 and Px_2, respectively ($PQ = \mathrm{d}x_1$, $PR = \mathrm{d}x_2$). In the deformation, they go to points Q' and R' defined by

$$\mathbf{QQ}': \begin{cases} \mathrm{d}x'_1 = \mathrm{d}x_1 + (\partial u_1/\partial x_1)\mathrm{d}x_1 \\ \mathrm{d}x'_2 = (\partial u_2/\partial x_1)\mathrm{d}x_1 \\ \mathrm{d}x'_3 = 0 \end{cases}$$

$$\mathbf{RR}': \begin{cases} \mathrm{d}x'_1 = (\partial u_1/\partial x_2)\mathrm{d}x_2 \\ \mathrm{d}x'_2 = \mathrm{d}x_2 + (\partial u_2/\partial x_2)\mathrm{d}x_2 \\ \mathrm{d}x'_3 = 0. \end{cases}$$

As the coefficients $\partial u_i/\partial x_j$ are small, the lengths of $P'Q'$ and $P'R'$ are hardly different from PQ and PR, respectively, and the elongations in the directions Px_1 and Px_2 are

$$\frac{P'Q' - PQ}{PQ} = \frac{\mathrm{d}x'_1 - \mathrm{d}x_1}{\mathrm{d}x_1} = \frac{\partial u_1}{\partial x_1} = S_1$$
$$\frac{P'R' - PR}{PR} = \frac{\mathrm{d}x'_2 - \mathrm{d}x_2}{\mathrm{d}x_2} = \frac{\partial u_2}{\partial x_2} = S_2.$$

The components S_1, S_2, S_3 of the principal diagonal of the Voigt matrix can then be interpreted as the elongations in the three directions Px_1, Px_2 and Px_3. The angles α and β between **PQ** and **P′Q′**, and **PR** and **P′R′**, respectively, are given in the same way by

$$\alpha = \mathrm{d}x'_2/\mathrm{d}x_1 = \partial u_2/\partial x_1; \quad \beta = \mathrm{d}x'_1/dx_2 = \partial u_1/\partial x_2.$$

One sees that the coefficient S_6 of Voigt's matrix is therefore

$$S_6 = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} = \alpha + \beta.$$

The angle $\alpha + \beta$ is equal to the difference between angles $\mathbf{PQ} \wedge \mathbf{PR}$ before deformation and $\mathbf{P''Q''} \wedge \mathbf{P'R'}$ after deformation. The nondiagonal terms of the Voigt matrix therefore represent the shears in the planes parallel to Px_1, Px_2 and Px_3, respectively.

To summarize, if one considers a small cube before deformation, it becomes after deformation an arbitrary parallelepiped; the relative elongations of the three sides are given by the diagonal terms of the strain tensor and the variation of the angles by its nondiagonal terms.

The cubic dilatation (1.3.1.5) is

$$\Delta(B) - 1 = S_1 + S_2 + S_3$$

(taking into account the fact that the coefficients S_{ij} are small).

1.3.1.4. *Particular components of the deformation*

1.3.1.4.1. *Simple elongation*

Matrix M has only one coefficient, e_1, and reduces to (Fig. 1.3.1.5*a*)

$$\begin{pmatrix} e_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The quadric of elongations is reduced to two parallel planes, perpendicular to Ox_1, with the equation $x_1 = \pm 1/\sqrt{|e|}$.

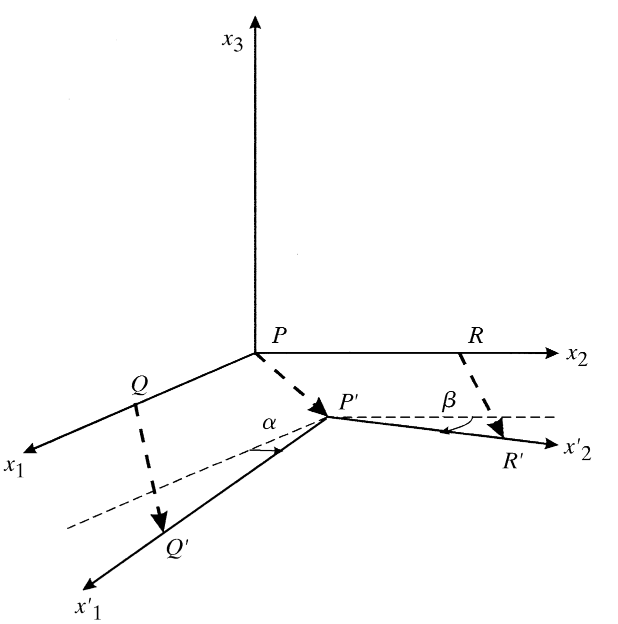

Fig. 1.3.1.4. Geometrical interpretation of the components of the strain tensor. Ox_1, Ox_2, Ox_3: axes before deformation; Ox'_1, Ox'_2, Ox'_3: axes after deformation.

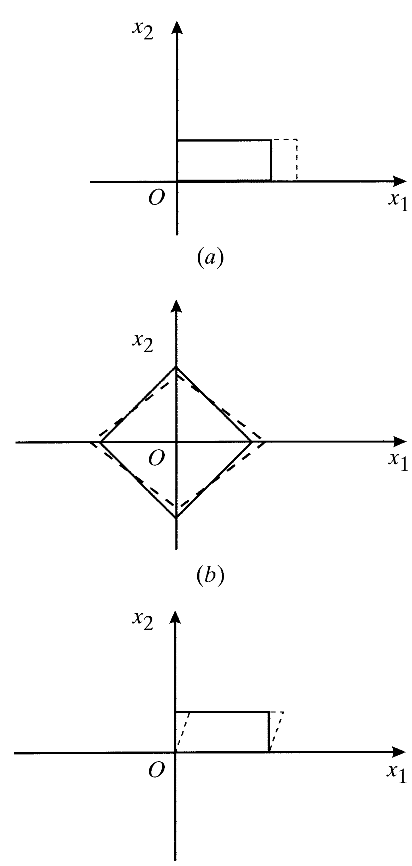

Fig. 1.3.1.5. Special deformations. The state after deformation is represented by a dashed line. (*a*) Simple elongation; (*b*) pure shear; (*c*) simple shear.

1.3.1.4.2. *Pure shear*

This is a pure deformation (without rotation) consisting of the superposition of two simple elongations along two perpendicular directions (Fig. 1.3.1.5*b*) and such that there is no change of volume (the cubic dilatation is zero):

$$\begin{pmatrix} e_1 & 0 & 0 \\ 0 & -e_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The quadric of elongations is a hyperbolic cylinder.

1.3.1.4.3. *Simple shear*

Matrix M_{ij} has one coefficient only, a shear (Fig. 1.3.1.5*c*):

$$\begin{pmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix is not symmetrical, as it contains a component of rotation. Thus we have

$$\left.\begin{aligned} x'_1 &= x_1 + sx_2 \\ x'_2 &= x_2 \\ x'_3 &= x_3. \end{aligned}\right\}$$

One can show that the deformation is a pure shear associated with a rotation around Ox_3.

1.3.2. Stress tensor

1.3.2.1. *General conditions of equilibrium of a solid*

Let us consider a solid C, in movement or not, with a mass distribution defined by a specific mass ρ at each point. There are two types of force that are manifested in the interior of this solid.

(i) *Body forces* (or mass forces), which one can write in the form

$$\mathbf{F}\,\mathrm{d}m = \mathbf{F}\rho\,\mathrm{d}\tau,$$

where $\mathrm{d}\tau$ is a volume element and $\mathrm{d}m$ a mass element. Gravity forces or inertial forces are examples of body forces. One can also envisage body torques (or volume couples), which can arise, for example, from magnetic or electric actions but which will be seen to be neglected in practice.

(ii) *Surface forces or stresses*. Let us imagine a cut in the solid along a surface element $\mathrm{d}\sigma$ of normal $\mathbf{n}$ (Fig. 1.3.2.1). The two lips of the cut that were in equilibrium are now subjected to equal and opposite forces, $\mathbf{R}$ and $\mathbf{R}' = -\mathbf{R}$, which will tend to separate or draw together these two lips. One admits that, when the area element $\mathrm{d}\sigma$ tends towards zero, the ratio $\mathbf{R}/\mathrm{d}\sigma$ tends towards a finite limit, $\mathbf{T}_n$, which is called *stress*. It is a force per unit area of surface, homogeneous to a pressure. It will be considered as positive if it is oriented towards the same side of the surface-area element $\mathrm{d}\sigma$ as the normal $\mathbf{n}$ and negative in the other case. The choice of the orientation of $\mathbf{n}$ is arbitrary. The pressure in a liquid is defined in a similar way but its magnitude is independent of the orientation of $\mathbf{n}$ and its direction is always parallel to $\mathbf{n}$. On the other hand, in a solid the constraint $\mathbf{T}_n$ applied to a surface element is not necessarily normal to the latter and the magnitude and the orientation with respect to the normal change when the orientation of $\mathbf{n}$ changes. A stress is said to be *homogeneous* if the force per unit area acting on a surface element of given orientation and given shape is independent of the position of the element in the body. Other stresses are *inhomogeneous*. Pressure is represented by a scalar, and stress by a rank-two tensor, which will be defined in Section 1.3.2.2.

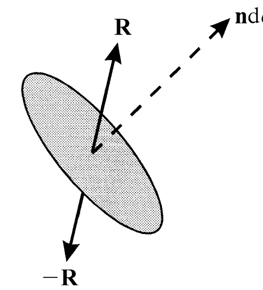

Fig. 1.3.2.1. Definition of stress: it is the limit of $\mathbf{R}$ $\mathrm{d}\sigma$ when the surface element $\mathrm{d}\sigma$ tends towards zero. $\mathbf{R}$ and $\mathbf{R}'$ are the forces to which the two lips of the small surface element cut within the medium are subjected.

Now consider a volume V within the solid C and the surface S which surrounds it (Fig. 1.3.2.2). Among the influences that are exterior to V, we distinguish those that are external to the solid C and those that are internal. The first are translated by the body forces, eventually by volume couples. The second are translated by the local contact forces of the part external to V on the internal part; they are represented by a surface density of forces, *i.e.* by the stresses $\mathbf{T}_n$ that depend only on the point Q of the surface S where they are applied and on the orientation of the normal $\mathbf{n}$ of this surface at this point. If two surfaces S and S' are tangents at the same point Q, the same stress acts at the point of contact between them. The equilibrium of the volume V requires:

(i) For the resultant of the applied forces and the inertial forces:

$$\iint_S \mathbf{T}_n\,\mathrm{d}\sigma + \iiint_V \mathbf{F}\rho\,\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\iiint_V \mathbf{v}\,\mathrm{d}\tau\right\}. \qquad (1.3.2.1)$$

(ii) For the resultant moment:

$$\iint_S \mathbf{OQ}\wedge\mathbf{T}_n\,\mathrm{d}\sigma + \iiint_V \mathbf{OP}\wedge\mathbf{F}\rho\,\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\iiint_V \mathbf{OP}\wedge\mathbf{v}\,\mathrm{d}\tau\right\}, \qquad (1.3.2.2)$$

where Q is a point on the surface S, P a point in the volume V and $\mathbf{v}$ the velocity of the volume element $\mathrm{d}\tau$.

The equilibrium of the solid C requires that:

(i) there are no stresses applied on its surface and

(ii) the above conditions are satisfied for *any* volume V within the solid C.

1.3.2.2. *Definition of the stress tensor*

Using the condition on the resultant of forces, it is possible to show that the components of the stress $\mathbf{T}_n$ can be determined from the knowledge of the orientation of the normal $\mathbf{n}$ and of the components of a rank-two tensor. Let P be a point situated inside volume V, Px_1, Px_2 and Px_3 three orthonormal axes, and consider a plane of arbitrary orientation that cuts the three axes at Q, R

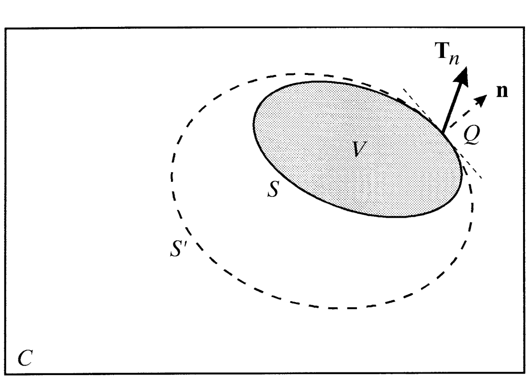

Fig. 1.3.2.2. Stress, $\mathbf{T}_n$, applied to the surface of an internal volume.

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