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whether the direction under consideration is outside or inside the asymptotic cone and intersects one or the other of the two hyperboloids.

Equally, one can connect the displacement vector $\mathbf{u}(\mathbf{r})$ directly with the quadric Q. Using the bilinear form

$$f(\mathbf{y}) = M_{ij} y_i y_j,$$

the gradient of $f(\mathbf{y})$, $\nabla(f)$, has as components

$$\partial f / \partial y^i = M_{ij} y_j = u_i.$$

One recognizes the components of the displacement vector $\mathbf{u}$, which is therefore parallel to the normal to the quadric Q at the extremity of the radius vector $\mathbf{Op}$ parallel to $\mathbf{r}$.

The directions of the principal axes of Q correspond to the extremal values of y, *i.e.* to the stationary values (maximal or minimal) of the elongation. These values are the *principal elongations*.

If the deformation is a pure rotation

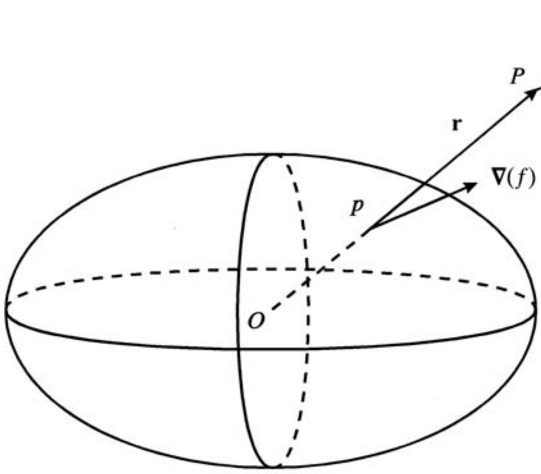
$$B = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M = \begin{pmatrix} \cos\theta - 1 & \sin\theta & 0 \\ -\sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence we have

$$M_{ij} y_i y_j = (\cos\theta - 1)(y_1 - y_2) = \varepsilon.$$

The quadric Q is a cylinder of revolution having the axis of rotation as axis.



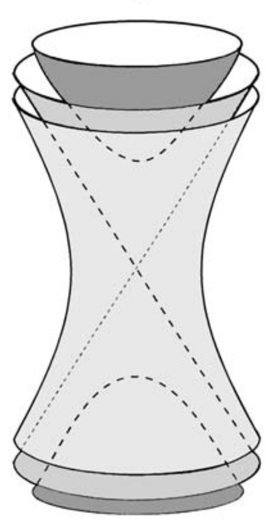


Fig. 1.3.1.3. Quadric of elongations. The displacement vector, $\mathbf{u}(\mathbf{r})$, at P in the deformed medium is parallel to the normal to the quadric at the intersection, p, of $\mathbf{OP}$ with the quadric. (a) The eigenvalues all have the same sign, the quadric is an ellipsoid. (b) The eigenvalues have mixed signs, the quadric is a hyperboloid with either one sheet (shaded in light grey) or two sheets (shaded in dark grey), depending on the sign of the constant ε [see equation (1.3.1.7)]; the cone asymptote is represented in medium grey. For a practical application, see Fig. 1.4.1.1.

1.3.1.3. *Arbitrary but small deformations*

1.3.1.3.1. *Definition of the strain tensor*

If the deformation is small but arbitrary, *i.e.* if the products of two or more components of M_{ij} can be neglected with respect to unity, one can describe the deformation locally as a homogeneous asymptotic deformation. As was shown in Section 1.3.1.2.4, it can be put in the form of the product of a pure deformation corresponding to the symmetric part of M_{ij}, S_{ij}, and a pure rotation corresponding to the asymmetric part, A_{ij}:

$$\left.\begin{aligned} S_{ij} &= S_{ji} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\ A_{ij} &= -A_{ji} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \end{aligned}\right\} \qquad (1.3.1.8)$$

Matrix B can be written

$$B = I + A + S,$$

where I is the matrix identity. As the coefficients $\partial u_i / \partial x_j$ of M_{ij} are small, one can neglect the product $A \times S$ and one has

$$B = (I + A)(I + S).$$

$(I + S)$ is a symmetric matrix that represents a pure deformation. $(I + A)$ is an antisymmetric unitary matrix and, since A is small,

$$(I + A)^{-1} = (I - A).$$

Thus, $(I + A)$ represents a rotation. The axis of rotation is parallel to the vector with coordinates

$$\left.\begin{aligned} \Omega_1 &= \tfrac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) = A_{32} \\ \Omega_2 &= \tfrac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = A_{13} \\ \Omega_3 &= \tfrac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) = A_{21}, \end{aligned}\right\}$$

which is an eigenvector of $(I + A)$. The magnitude of the rotation is equal to the modulus of this vector.

In general, one is only interested in the pure deformation, *i.e.* in the form of the deformed object. Thus, one only wishes to know the quantities $(I + S)$ and the symmetric part of M. It is this symmetric part that is called the deformation tensor or the strain tensor. It is very convenient for applications to use the simplified notation due to Voigt:

$$S_1 = \frac{\partial u_1}{\partial x_1}; \qquad S_2 = \frac{\partial u_2}{\partial x_2}; \qquad S_3 = \frac{\partial u_3}{\partial x_3};$$

$$S_4 = \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}; \quad S_5 = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}; \quad S_6 = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}.$$

One may note that

$$S_1 = S_{11}; \quad S_2 = S_{22}; \quad S_3 = S_{33};$$
$$S_4 = S_{23} + S_{32}; \quad S_5 = S_{31} + S_{13}; \quad S_6 = S_{12} + S_{21}.$$

The Voigt strain matrix S is of the form

$$\begin{pmatrix} S_1 & S_6 & S_5 \\ S_6 & S_2 & S_4 \\ S_5 & S_4 & S_3 \end{pmatrix}.$$

1.3.1.3.2. *Geometrical interpretation of the coefficients of the strain tensor*

Let us consider an orthonormal system of axes with centre P. We remove nothing from the generality of the following by limiting ourselves to a planar problem and assuming that point P' to which P goes in the deformation lies in the plane x_1Px_2 (Fig. 1.3.1.4). Let us consider two neighbouring points, Q and R, lying on axes Px_1 and Px_2, respectively ($PQ = \mathrm{d}x_1$, $PR = \mathrm{d}x_2$). In the deformation, they go to points Q' and R' defined by

$$\mathbf{QQ}': \begin{cases} \mathrm{d}x'_1 = \mathrm{d}x_1 + (\partial u_1/\partial x_1)\mathrm{d}x_1 \\ \mathrm{d}x'_2 = (\partial u_2/\partial x_1)\mathrm{d}x_1 \\ \mathrm{d}x'_3 = 0 \end{cases}$$

$$\mathbf{RR}': \begin{cases} \mathrm{d}x'_1 = (\partial u_1/\partial x_2)\mathrm{d}x_2 \\ \mathrm{d}x'_2 = \mathrm{d}x_2 + (\partial u_2/\partial x_2)\mathrm{d}x_2 \\ \mathrm{d}x'_3 = 0. \end{cases}$$

As the coefficients $\partial u_i/\partial x_j$ are small, the lengths of $P'Q'$ and $P'R'$ are hardly different from PQ and PR, respectively, and the elongations in the directions Px_1 and Px_2 are

$$\frac{P'Q' - PQ}{PQ} = \frac{\mathrm{d}x'_1 - \mathrm{d}x_1}{\mathrm{d}x_1} = \frac{\partial u_1}{\partial x_1} = S_1$$
$$\frac{P'R' - PR}{PR} = \frac{\mathrm{d}x'_2 - \mathrm{d}x_2}{\mathrm{d}x_2} = \frac{\partial u_2}{\partial x_2} = S_2.$$

The components S_1, S_2, S_3 of the principal diagonal of the Voigt matrix can then be interpreted as the elongations in the three directions Px_1, Px_2 and Px_3. The angles α and β between **PQ** and **P′Q′**, and **PR** and **P′R′**, respectively, are given in the same way by

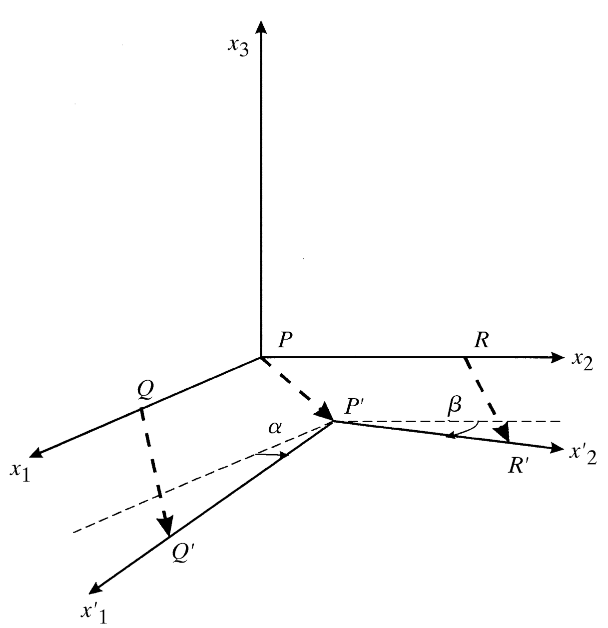


Fig. 1.3.1.4. Geometrical interpretation of the components of the strain tensor. Ox_1, Ox_2, Ox_3: axes before deformation; Ox'_1, Ox'_2, Ox'_3: axes after deformation.

$$\alpha = \mathrm{d}x'_2/\mathrm{d}x_1 = \partial u_2/\partial x_1; \quad \beta = \mathrm{d}x'_1/dx_2 = \partial u_1/\partial x_2.$$

One sees that the coefficient S_6 of Voigt's matrix is therefore

$$S_6 = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} = \alpha + \beta.$$

The angle $\alpha + \beta$ is equal to the difference between angles $\mathbf{PQ} \wedge \mathbf{PR}$ before deformation and $\mathbf{P''Q''} \wedge \mathbf{P'R'}$ after deformation. The nondiagonal terms of the Voigt matrix therefore represent the shears in the planes parallel to Px_1, Px_2 and Px_3, respectively.

To summarize, if one considers a small cube before deformation, it becomes after deformation an arbitrary parallelepiped; the relative elongations of the three sides are given by the diagonal terms of the strain tensor and the variation of the angles by its nondiagonal terms.

The cubic dilatation (1.3.1.5) is

$$\Delta(B) - 1 = S_1 + S_2 + S_3$$

(taking into account the fact that the coefficients S_{ij} are small).

1.3.1.4. *Particular components of the deformation*

1.3.1.4.1. *Simple elongation*

Matrix M has only one coefficient, e_1, and reduces to (Fig. 1.3.1.5*a*)

$$\begin{pmatrix} e_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The quadric of elongations is reduced to two parallel planes, perpendicular to Ox_1, with the equation $x_1 = \pm 1/\sqrt{|e|}$.

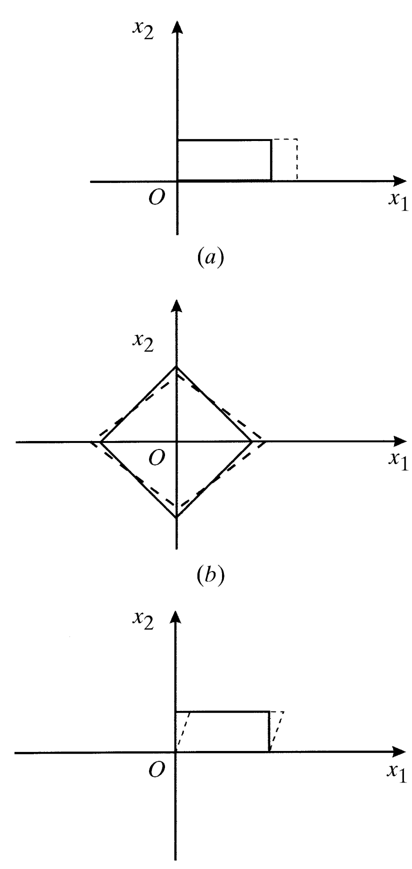


Fig. 1.3.1.5. Special deformations. The state after deformation is represented by a dashed line. (*a*) Simple elongation; (*b*) pure shear; (*c*) simple shear.

references