International Tables for Crystallography (2013). Vol. D, Section 1.3.2, pp. 76–80.

1.3.1.4.2. *Pure shear*

This is a pure deformation (without rotation) consisting of the superposition of two simple elongations along two perpendicular directions (Fig. 1.3.1.5*b*) and such that there is no change of volume (the cubic dilatation is zero):

$$\begin{pmatrix} e_1 & 0 & 0 \\ 0 & -e_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The quadric of elongations is a hyperbolic cylinder.

1.3.1.4.3. *Simple shear*

Matrix M_{ij} has one coefficient only, a shear (Fig. 1.3.1.5*c*):

$$\begin{pmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix is not symmetrical, as it contains a component of rotation. Thus we have

$$\left.\begin{aligned} x'_1 &= x_1 + s x_2 \\ x'_2 &= x_2 \\ x'_3 &= x_3. \end{aligned}\right\}$$

One can show that the deformation is a pure shear associated with a rotation around Ox_3.

1.3.2. Stress tensor

1.3.2.1. *General conditions of equilibrium of a solid*

Let us consider a solid C, in movement or not, with a mass distribution defined by a specific mass ρ at each point. There are two types of force that are manifested in the interior of this solid.

(i) *Body forces* (or mass forces), which one can write in the form

$$\mathbf{F}\,\mathrm{d}m = \mathbf{F}\rho\,\mathrm{d}\tau,$$

where $\mathrm{d}\tau$ is a volume element and $\mathrm{d}m$ a mass element. Gravity forces or inertial forces are examples of body forces. One can also envisage body torques (or volume couples), which can arise, for example, from magnetic or electric actions but which will be seen to be neglected in practice.

(ii) *Surface forces or stresses*. Let us imagine a cut in the solid along a surface element $\mathrm{d}\sigma$ of normal $\mathbf{n}$ (Fig. 1.3.2.1). The two lips of the cut that were in equilibrium are now subjected to equal and opposite forces, $\mathbf{R}$ and $\mathbf{R}' = -\mathbf{R}$, which will tend to separate or draw together these two lips. One admits that, when the area element $\mathrm{d}\sigma$ tends towards zero, the ratio $\mathbf{R}/\mathrm{d}\sigma$ tends towards a finite limit, $\mathbf{T}_n$, which is called *stress*. It is a force per unit area of surface, homogeneous to a pressure. It will be considered as positive if it is oriented towards the same side of the surface-area element $\mathrm{d}\sigma$ as the normal $\mathbf{n}$ and negative in the other case. The choice of the orientation of $\mathbf{n}$ is arbitrary. The pressure in a liquid is defined in a similar way but its magnitude is independent of the orientation of $\mathbf{n}$ and its direction is always parallel to $\mathbf{n}$. On the other hand, in a solid the constraint $\mathbf{T}_n$ applied to a surface element is not necessarily normal to the latter and the magnitude and the orientation with respect to the normal change when the orientation of $\mathbf{n}$ changes. A stress is said to be *homogeneous* if the force per unit area acting on a surface element of given orientation and given shape is independent of the position of the element in the body. Other stresses are *inhomogeneous*. Pressure is represented by a scalar, and stress by a rank-two tensor, which will be defined in Section 1.3.2.2.

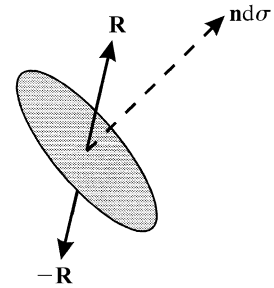

Fig. 1.3.2.1. Definition of stress: it is the limit of $\mathbf{R}$ $\mathrm{d}\sigma$ when the surface element $\mathrm{d}\sigma$ tends towards zero. $\mathbf{R}$ and $\mathbf{R}'$ are the forces to which the two lips of the small surface element cut within the medium are subjected.

Now consider a volume V within the solid C and the surface S which surrounds it (Fig. 1.3.2.2). Among the influences that are exterior to V, we distinguish those that are external to the solid C and those that are internal. The first are translated by the body forces, eventually by volume couples. The second are translated by the local contact forces of the part external to V on the internal part; they are represented by a surface density of forces, *i.e.* by the stresses $\mathbf{T}_n$ that depend only on the point Q of the surface S where they are applied and on the orientation of the normal $\mathbf{n}$ of this surface at this point. If two surfaces S and S' are tangents at the same point Q, the same stress acts at the point of contact between them. The equilibrium of the volume V requires:

(i) For the resultant of the applied forces and the inertial forces:

$$\iint_S \mathbf{T}_n\,\mathrm{d}\sigma + \iiint_V \mathbf{F}\rho\,\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\iiint_V \mathbf{v}\,\mathrm{d}\tau\right\}. \qquad (1.3.2.1)$$

(ii) For the resultant moment:

$$\iint_S \mathbf{OQ}\wedge\mathbf{T}_n\,\mathrm{d}\sigma + \iiint_V \mathbf{OP}\wedge\mathbf{F}\rho\,\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\iiint_V \mathbf{OP}\wedge\mathbf{v}\,\mathrm{d}\tau\right\}, \qquad (1.3.2.2)$$

where Q is a point on the surface S, P a point in the volume V and $\mathbf{v}$ the velocity of the volume element $\mathrm{d}\tau$.

The equilibrium of the solid C requires that:

(i) there are no stresses applied on its surface and

(ii) the above conditions are satisfied for *any* volume V within the solid C.

1.3.2.2. *Definition of the stress tensor*

Using the condition on the resultant of forces, it is possible to show that the components of the stress $\mathbf{T}_n$ can be determined from the knowledge of the orientation of the normal $\mathbf{n}$ and of the components of a rank-two tensor. Let P be a point situated inside volume V, Px_1, Px_2 and Px_3 three orthonormal axes, and consider a plane of arbitrary orientation that cuts the three axes at Q, R

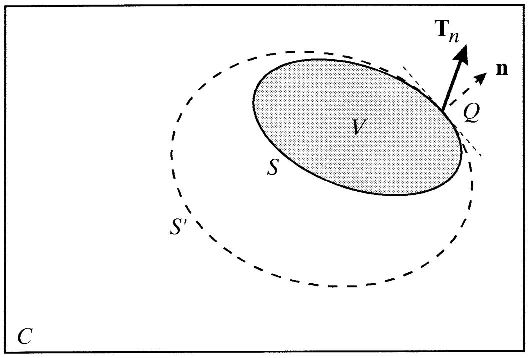

Fig. 1.3.2.2. Stress, $\mathbf{T}_n$, applied to the surface of an internal volume.

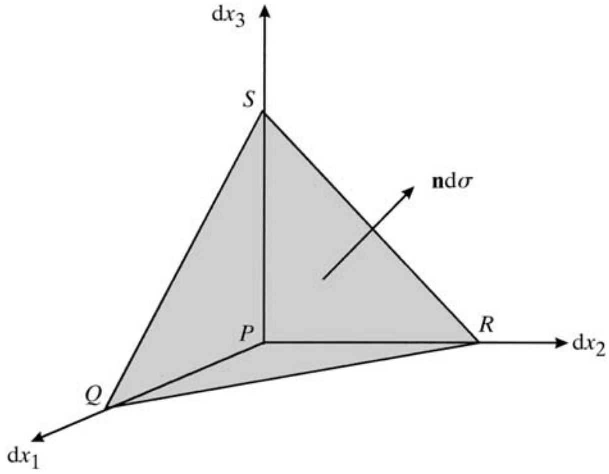

Fig. 1.3.2.3. Equilibrium of a small volume element.

and S, respectively (Fig. 1.3.2.3). The small volume element $PQRS$ is limited by four surfaces to which stresses are applied. The normals to the surfaces PRS, PSQ and PQR will be assumed to be directed towards the interior of the small volume. By contrast, for reasons that will become apparent later, the normal $\mathbf{n}$ applied to the surface QRS will be oriented towards the *exterior*. The corresponding applied forces are thus given in Table 1.3.2.1. The volume $PQRS$ is subjected to five forces: the forces applied to each surface and the resultant of the volume forces and the inertial forces. The equilibrium of the small volume requires that the resultant of these forces be equal to zero and one can write

$$-\mathbf{T}_n \,\mathrm{d}\sigma + \mathbf{T}_1 \,\mathrm{d}\sigma_1 + \mathbf{T}_2 \,\mathrm{d}\sigma_2 + \mathbf{T}_3 \,\mathrm{d}\sigma_3 + \mathbf{F}\rho \,\mathrm{d}\tau = 0$$

(including the inertial forces in the volume forces).

As long as the surface element $\mathrm{d}\sigma$ is finite, however small, it is possible to divide both terms of the equation by it. If one introduces the direction cosines, α_i, the equation becomes

$$-\mathbf{T}_n + \mathbf{T}_1 \,\mathrm{d}\alpha_1 + \mathbf{T}_2 \,\mathrm{d}\alpha_2 + \mathbf{T}_3 \,\mathrm{d}\alpha_3 + \mathbf{F}\rho \,\mathrm{d}\tau/\mathrm{d}\sigma = 0.$$

When $\mathrm{d}\sigma$ tends to zero, the ratio $\mathrm{d}\sigma/\mathrm{d}\tau$ tends towards zero at the same time and may be neglected. The relation then becomes

$$\mathbf{T}_n = \mathbf{T}_i\alpha^i. \qquad (1.3.2.3)$$

This relation is called the Cauchy relation, which allows the stress $\mathbf{T}_n$ to be expressed as a function of the stresses $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$ that are applied to the three faces perpendicular to the axes, Px_1, Px_2 and Px_3. Let us project this relation onto these three axes:

$$T_{nj} = T_{ij}\alpha_i. \qquad (1.3.2.4)$$

The nine components T_{ij} are, by definition, the components of the stress tensor. In order to check that they are indeed the components of a tensor, it suffices to make the contracted product of each side of (1.3.2.4) by any vector x_i: the left-hand side is a scalar product and the right-hand side a bilinear form. The T_{ij}'s are therefore the components of a tensor. The index to the far left indicates the face to which the stress is applied (normal to the x_1, x_2 or x_3 axis), while the second one indicates on which axis the stress is projected.

Table 1.3.2.1. *Stresses applied to the faces surrounding a volume element*

α_1, α_2 and α_3 are the direction cosines of the normal $\mathbf{n}$ to the small surface QRS.

Face	Area	Applied stress	Applied force
QRS	$\mathrm{d}\sigma$	$-\mathbf{T}_n$	$-\mathbf{T}_n \,\mathrm{d}\sigma$
PRS	$\mathrm{d}\sigma_1 = \alpha_1 \,\mathrm{d}\sigma$	$\mathbf{T}_1$	$\mathbf{T}_n \,\mathrm{d}\sigma_1$
PSQ	$\mathrm{d}\sigma_2 = \alpha_2 \,\mathrm{d}\sigma$	$\mathbf{T}_2$	$\mathbf{T}_n \,\mathrm{d}\sigma_2$
PQR	$\mathrm{d}\sigma_3 = \alpha_3 \,\mathrm{d}\sigma$	$\mathbf{T}_3$	$\mathbf{T}_n \,\mathrm{d}\sigma_3$

1.3.2.3. *Condition of continuity*

Let us return to equation (1.3.2.1) expressing the equilibrium condition for the resultant of the forces. By replacing $\mathbf{T}_n$ by the expression (1.3.2.4), we get, after projection on the three axes,

$$\int\!\!\int_S T_{ij} \,\mathrm{d}\sigma_i + \int\!\!\int\!\!\int_V F_j\rho \,\mathrm{d}\tau = 0,$$

where $\mathrm{d}\sigma_i = \alpha_i \,\mathrm{d}\sigma$ and the inertial forces are included in the volume forces. Applying Green's theorem to the first integral, we have

$$\int\!\!\int_S T_{ij} \,\mathrm{d}\sigma_i = \int\!\!\int\!\!\int_V \left[\partial T_{ij}/\partial x_i\right]\mathrm{d}\tau.$$

The equilibrium condition now becomes

$$\int\!\!\int\!\!\int_V \left[\partial T_{ij}/\partial x_i + F_j\rho\right]\mathrm{d}\tau = 0.$$

In order that this relation applies to any volume V, the expression under the integral must be equal to zero,

$$\partial T_{ij}/\partial x_i + F_j\rho = 0, \qquad (1.3.2.5)$$

or, if one includes explicitly the inertial forces,

$$\partial T_{ij}/\partial x_i + F_j\rho = -\rho \,\partial^2 x_j/\partial t^2. \qquad (1.3.2.6)$$

This is the condition of continuity or of conservation. It expresses how constraints propagate throughout the solid. This is how the cohesion of the solid is ensured. The resolution of any elastic problem requires solving this equation in terms of the particular boundary conditions of that problem.

1.3.2.4. *Symmetry of the stress tensor*

Let us now consider the equilibrium condition (1.3.2.2) relative to the resultant moment. After projection on the three axes, and using the Cartesian expression (1.1.3.4) of the vectorial products, we obtain

$$\int\!\!\int_S \tfrac{1}{2}\varepsilon_{ijk}(x_iT_{lj} - x_jT_{li})\mathrm{d}\sigma_l + \int\!\!\int\!\!\int_V \left[\tfrac{1}{2}\varepsilon_{ijk}\rho(x_iF_j - x_jF_i) + \Gamma_k\right]\mathrm{d}\tau = 0.$$

(including the inertial forces in the volume forces). ε_{ijk} is the permutation tensor. Applying Green's theorem to the first integral and putting the two terms together gives

$$\int\!\!\int\!\!\int_V \left\{\tfrac{1}{2}\varepsilon_{ijk}\left[\frac{\partial}{\partial x_l}(x_iT_{lj} - x_jT_{li}) + \rho(x_iF_j - x_jF_i)\right] + \Gamma_k\right\}\mathrm{d}\tau = 0.$$

In order that this relation applies to any volume V within the solid C, we must have

$$\tfrac{1}{2}\varepsilon_{ijk}\left[\frac{\partial}{\partial x_l}(x_iT_{lj} - x_jT_{li})\right] + \Gamma_k = 0$$

or

$$\tfrac{1}{2}\varepsilon_{ijk}\left[x_i\left(\frac{\partial T_{lj}}{\partial x_l} + F_j\rho\right) - x_j\left(\frac{\partial T_{li}}{\partial x_l} + F_i\rho\right) + T_{ij} - T_{ji}\right] + \Gamma_k = 0.$$

Taking into account the continuity condition (1.3.2.5), this equation reduces to

$$\tfrac{1}{2}\varepsilon_{ijk}[T_{ij} - T_{ji}] + \Gamma_k = 0.$$

A volume couple can occur for instance in the case of a magnetic or an electric field acting on a body that locally possesses magnetic or electric moments. In general, apart from very rare cases, one can ignore these volume couples. One can then deduce that the stress tensor is symmetrical:

$$T_{ij} - T_{ji} = 0.$$

This result can be recovered by applying the relation (1.3.2.2) to a small volume in the form of an elementary parallelepiped, thus illustrating the demonstration using Green's theorem but giving insight into the action of the constraints. Consider a rectangular parallelepiped, of sides $2\Delta x_1$, $2\Delta x_2$ and $2\Delta x_3$, with centre P at the origin of an orthonormal system whose axes Px_1, Px_2 and Px_3 are normal to the sides of the parallelepiped (Fig. 1.3.2.4). In order that the resultant moment with respect to a point be zero, it is necessary that the resultant moments with respect to three axes concurrent in this point are zero. Let us write for instance that the resultant moment with respect to the axis Px_3 is zero. We note that the constraints applied to the faces perpendicular to Px_3 do not give rise to a moment and neither do the components T_{11}, T_{13}, T_{22} and T_{23} of the constraints applied to the faces normal to Px_1 and Px_2 (Fig. 1.3.2.4). The components T_{12} and T_{21} alone have a nonzero moment.

For face 1, the constraint is $T_{12} + (\partial T_{12}/\partial x_1)\Delta x_1$ if T_{12} is the magnitude of the constraint at P. The force applied at face 1 is

$$\left[T_{12} + \frac{\partial T_{12}}{\partial x_1}\Delta x_1\right] 4\Delta x_2 \Delta x_3$$

and its moment is

$$\left[T_{12} + \frac{\partial T_{12}}{\partial x_1}\Delta x_1\right] 4\Delta x_2 \Delta x_3 \Delta x_1.$$

Similarly, the moments of the force on the other faces are

$$\text{Face } 1': \quad -\left[T_{12} + \frac{\partial T_{12}}{\partial x_1}(-\Delta x_1)\right] 4\Delta x_2 \Delta x_3(-\Delta x_1);$$

$$\text{Face } 2: \quad \left[T_{21} + \frac{\partial T_{21}}{\partial x_2}\Delta x_2)\right] 4\Delta x_1 \Delta x_3 \Delta x_2;$$

$$\text{Face } 2': \quad -\left[T_{21} + \frac{\partial T_{21}}{\partial x_2}(-\Delta x_2)\right] 4\Delta x_1 \Delta x_3(-\Delta x_2).$$

Noting further that the moments applied to the faces 1 and 1′ are of the same sense, and that those applied to faces 2 and 2′ are of the opposite sense, we can state that the resultant moment is

$$[T_{12} - T_{21}]8\Delta x_1 \Delta x_2 \Delta x_3 = [T_{12} - T_{21}]\Delta\tau,$$

where $8\Delta x_1 \Delta x_2 \Delta x_3 = \Delta\tau$ is the volume of the small parallelepiped. The resultant moment per unit volume, taking into account the couples in volume, is therefore

$$T_{12} - T_{21} + \Gamma_3.$$

It must equal zero and the relation given above is thus recovered.

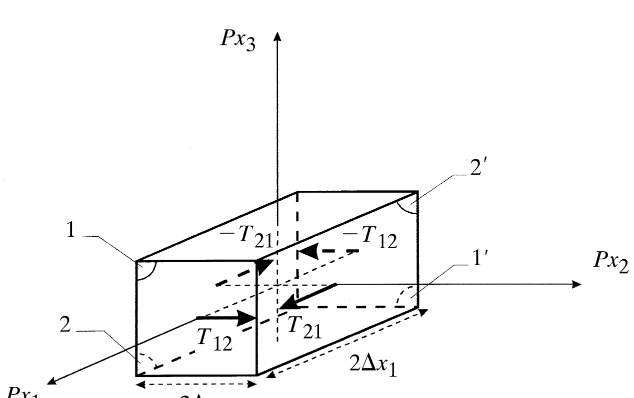

Fig. 1.3.2.4. Symmetry of the stress tensor: the moments of the couples applied to a parallelepiped compensate each other.

1.3.2.5. *Voigt's notation – interpretation of the components of the stress tensor*

1.3.2.5.1. *Voigt's notation, reduced form of the stress tensor*

We shall use frequently the notation due to Voigt (1910) in order to express the components of the stress tensor:

$$T_1 = T_{11}; \quad T_2 = T_{22}; \quad T_3 = T_{33};$$
$$T_4 = T_{23} = T_{32}; \quad T_5 = T_{31} = T_{13}; \quad T_6 = T_{12} = T_{21}.$$

It should be noted that the conventions are different for the Voigt matrices associated with the stress tensor and with the strain tensor (Section 1.3.1.3.1).

The Voigt matrix associated with the stress tensor is therefore of the form

$$\begin{pmatrix} T_1 & T_6 & T_5 \\ T_6 & T_2 & T_4 \\ T_5 & T_4 & T_3 \end{pmatrix}.$$

1.3.2.5.2. *Interpretation of the components of the stress tensor – special forms of the stress tensor*

(i) *Uniaxial stress*: let us consider a solid shaped like a parallelepiped whose faces are normal to three orthonormal axes (Fig. 1.3.2.5). The terms of the main diagonal of the stress tensor correspond to uniaxial stresses on these faces. If there is a single uniaxial stress, the tensor is of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T_3 \end{pmatrix}.$$

The solid is submitted to two equal and opposite forces, $T_{33}S_3$ and $-T_{33}S_3$, where S_3 is the area of the face of the parallelepiped that is normal to the Ox_3 axis (Fig. 1.3.2.5*a*). The convention used in general is that there is a uniaxial *compression* if $T_3 \leq 0$ and a uniaxial *traction* if $T_3 \geq 0$, but the opposite sign convention is sometimes used, for instance in applications such as piezoelectricity or photoelasticity.

(ii) *Pure shear stress*: the tensor reduces to two equal uniaxial constraints of opposite signs (Fig. 1.3.2.5*b*):

$$\begin{pmatrix} T_1 & 0 & 0 \\ 0 & -T_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(iii) *Hydrostatic pressure*: the tensor reduces to three equal uniaxial stresses of the same sign (it is spherical):

$$\begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix},$$

where p is a positive scalar.

(iv) *Simple shear stress*: the tensor reduces to two equal nondiagonal terms (Fig. 1.3.2.5*c*), for instance $T_{12} = T_{21} = T_6$. T_{12} represents the component parallel to Ox_2 of the stress applied to face 1 and T_{21} represents the component parallel to Ox_1 of the stress applied to face 2. These two stresses generate opposite couples that compensate each other. It is important to note that it is impossible to have one nondiagonal term only: its effect would be a couple of rotation of the solid and not a deformation.

1.3.2.6. *Boundary conditions*

If the surface of the solid C is free from all exterior action and is in equilibrium, the stress field T_{ij} inside C is zero at the surface. If C is subjected from the outside to a distribution of stresses T_n

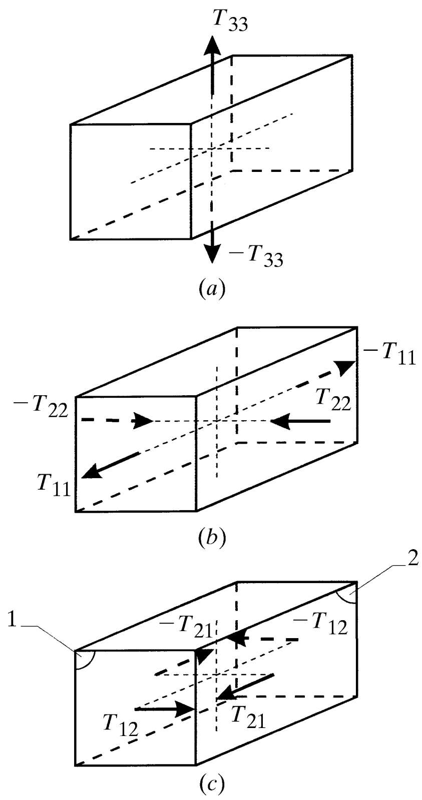

Fig. 1.3.2.5. Special forms of the stress tensor. (*a*) Uniaxial stress: the stress tensor has only one component, T_{33}; (*b*) pure shear stress: $T_{22} = -T_{11}$; (*c*) simple shear stress: $T_{21} = T_{12}$.

(apart from the volume forces mentioned earlier), the stress field inside the solid is such that at each point of the surface

$$T_{nj} = T_{ij}\alpha_i,$$

where the α_j's are the direction cosines of the normal to the surface at the point under consideration.

1.3.2.7. *Local properties of the stress tensor*

(i) *Normal stress and shearing stress*: let us consider a surface area element $d\sigma$ within the solid, the normal $\mathbf{n}$ to this element and the stress $\mathbf{T}_n$ that is applied to it (Fig. 1.3.2.6).

The *normal stress*, $\boldsymbol{\nu}$, is, by definition, the component of $\mathbf{T}_n$ on $\mathbf{n}$,

$$\boldsymbol{\nu} = \mathbf{n}(\mathbf{T}_n \cdot \mathbf{n})$$

and the *shearing stress*, $\boldsymbol{\tau}$, is the projection of $\mathbf{T}_n$ on the surface area element,

$$\boldsymbol{\tau} = \mathbf{n} \wedge (\mathbf{T}_n \wedge \mathbf{n}) = \mathbf{T}_n - \boldsymbol{\nu}.$$

(ii) *The stress quadric*: let us consider the bilinear form attached to the stress tensor:

$$f(\mathbf{y}) = T_{ij}y_i y_j.$$

The quadric represented by

$$f(\mathbf{y}) = \varepsilon$$

is called the stress quadric, where $\varepsilon = \pm 1$. It may be an ellipsoid or a hyperboloid. Referred to the principal axes, and using Voigt's notation, its equation is

$$y_i^2 T_i = \varepsilon.$$

To every direction $\mathbf{n}$ of the medium, let us associate the radius vector $\mathbf{y}$ of the quadric (Fig. 1.3.2.7) through the relation

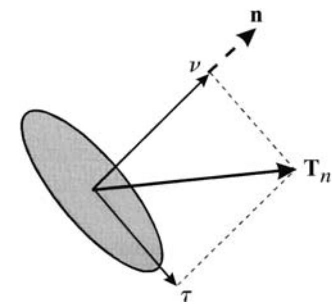

Fig. 1.3.2.6. Normal ($\boldsymbol{\nu}$) and shearing ($\boldsymbol{\tau}$) stress.

$$\mathbf{n} = k\mathbf{y}.$$

The stress applied to a small surface element $d\sigma$ normal to $\mathbf{n}$, $\mathbf{T}_n$, is

$$\mathbf{T}_n = k\nabla(f)$$

and the normal stress, ν, is

$$\nu = \alpha_i T_i = 1/y^2,$$

where the α_i's are the direction cosines of $\mathbf{n}$.

(iii) *Principal normal stresses*: the stress tensor is symmetrical and has therefore real eigenvectors. If we represent the tensor with reference to a system of axes parallel to its eigenvectors, it is put in the form

$$\begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix}.$$

T_1, T_2 and T_3 are the principal normal stresses. The mean normal stress, T, is defined by the relation

$$T = (T_1 + T_2 + T_3)/3$$

and is an invariant of the stress tensor.

1.3.2.8. *Energy density in a deformed medium*

Consider a medium that is subjected to a stress field T_{ij}. It has sustained a deformation indicated by the deformation tensor S. During this deformation, the forces of contact have performed work and the medium has accumulated a certain elastic energy W. The knowledge of the energy density thus acquired is useful for studying the properties of the elastic constants Let the medium deform from the deformation S_{ij} to the deformation $S_{ij} + \delta S_{ij}$ under the influence of the stress field and let us evaluate the work of each component of the effort. Consider a small elementary rectangular parallelepiped of sides $2\Delta x_1$, $2\Delta x_2$, $2\Delta x_3$ (Fig. 1.3.2.8). We shall limit our calculation to the components T_{11} and T_{12}, which are applied to the faces 1 and 1′, respectively.

In the deformation δS, the point P goes to the point P', defined by

$$\mathbf{PP}' = \mathbf{u}(\mathbf{r}).$$

A neighbouring point Q goes to Q' such that (Fig. 1.3.1.1)

$$\mathbf{PQ} = \Delta\mathbf{r}; \quad \mathbf{P}'\mathbf{Q}' = \delta\mathbf{r}'.$$

The coordinates of $\delta\mathbf{r}'$ are given by

$$\delta x_i' = \delta\Delta x_i + \delta S_{ij}\delta x_j.$$

Of sole importance is the relative displacement of Q with respect to P and the displacement that must be taken into account in calculating the forces applied at Q. The coordinates of the relative displacement are

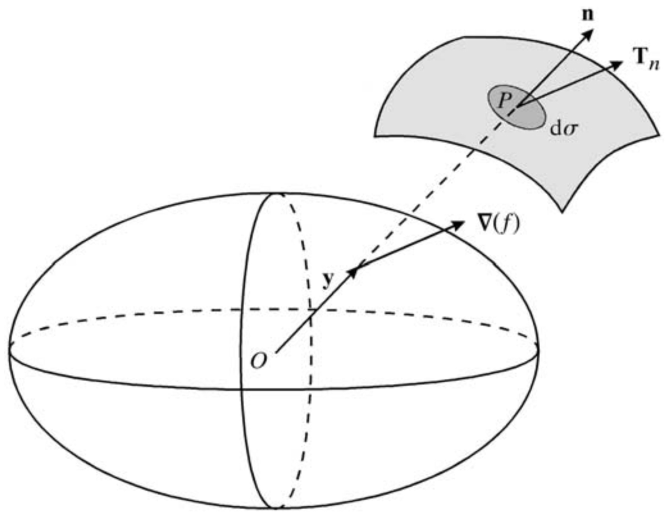

Fig. 1.3.2.7. The stress quadric: application to the determination of the stress applied to a surface element. The surface of the medium is shaded in light grey and a small surface element, dσ, is shaded in medium grey. The stress at P is proportional to $\nabla(f)$ at the intersection of **OP** with the stress quadric.

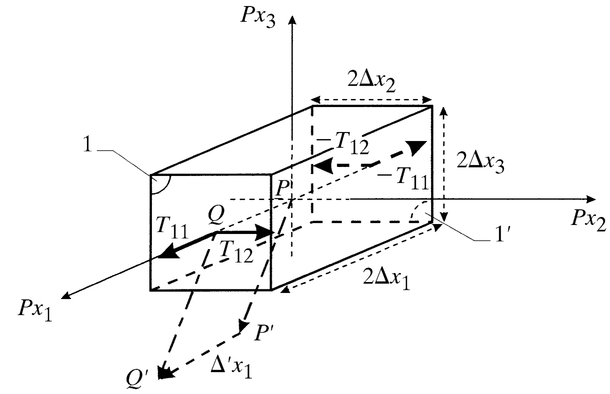

Fig. 1.3.2.8. Determination of the energy density in a deformed medium. **PP′** represents the displacement of the small parallelepiped during the deformation. The thick arrows represent the forces applied to the faces 1 and 1′.

$$\delta x'_i - \delta\Delta x_i = \delta S_{ij}\delta x_j.$$

We shall take as the position of Q the point of application of the forces at face 1, *i.e.* its centre with coordinates $\Delta x_1, 0, 0$ (Fig. 1.3.2.8). The area of face 1 is $4\Delta x_2 \Delta x_3$ and the forces arising from the stresses T_{11} and T_{12} are equal to $4\Delta x_2\Delta x_3 T_{11}$ and $4\Delta x_2 \Delta x_3 T_{12}$, respectively. The relative displacement of Q parallel to the line of action of T_{11} is $\Delta x_1 \delta S_{11}$ and the corresponding displacement along the line of action of T_{12} is $\Delta x_1 \delta S_{21}$. The work of the corresponding forces is therefore

$$\text{for } T_{11}: \quad 4\Delta x_1 \Delta x_2 \Delta x_3 T_{11} \delta S_{11}$$
$$\text{for } T_{12}: \quad 4\Delta x_1 \Delta x_2 \Delta x_3 T_{11} \delta S_{21}.$$

The work of the forces applied to the face 1′ is the same (T_{11}, T_{12} and x_1 change sign simultaneously). The works corresponding to the faces 1 and 1′ are thus $T_{11}\delta S_{11}\Delta\tau$ and $T_{12}\delta S_{21}\Delta\tau$ for the two stresses, respectively. One finds an analogous result for each of the other components of the stress tensor and the total work per unit volume is

$$\delta W = T_{ij}\delta S_{ji}. \qquad (1.3.2.7)$$

1.3.3. Linear elasticity

1.3.3.1. *Hooke's law*

Let us consider a metallic bar of length l_o loaded in pure tension (Fig. 1.3.3.1). Under the action of the uniaxial stress $T = F/\mathcal{A}$ (F applied force, $\mathcal{A}$ area of the section of the bar), the bar elongates and its length becomes $l = l_o + \Delta l$. Fig. 1.3.3.2 relates the variations of Δl and of the applied stress T. The curve representing the traction is very schematic and does not correspond to any real case. The following result, however, is common to all concrete situations:

(i) If $0 < T < T_o$, the deformation curve is reversible, *i.e.* if one releases the applied stress the bar resumes its original form. To a first approximation, the curve is linear, so that one can write *Hooke's law*:

$$\frac{\Delta l}{l} = \frac{1}{E} T, \qquad (1.3.3.1)$$

where E is the elastic stiffness, also called Young's modulus. The physical mechanism at the origin of elasticity is the deformation of the chemical bonds between atoms, ions or molecules in the solid, which act as so many small springs. The reaction of these springs to an applied stress is actually anharmonic and Hooke's law is only an approximation: a Taylor expansion up to the first term. A rigorous treatment of elasticity requires nonlinear phenomena to be taken into account. This is done in Section 1.3.6. The stress below which the strain is recoverable when the stress is removed, T_o, is called the *elastic limit*.

(ii) If $T > T_o$, the deformation curve is no longer reversible. If one releases the applied stress, the bar assumes a permanent deformation. One says that it has undergone a *plastic* deformation. The region of the deformation is ultimately limited by rupture (symbolized by an asterisk on Fig. 1.3.3.2). The plastic deformation is due to the formation and to the movement of lattice defects such as dislocations. The material in its initial state, before the application of a stress, is not free in general from defects and it possesses a complicated history of deformations. The proportionality constant between stresses and deformations in the elastic region depends on the interatomic force constants and is an intrinsic property, very little affected by the presence of defects. By contrast, the limit, T_o, of the elastic region depends to a large extent on the defects in the material and on its history. It is

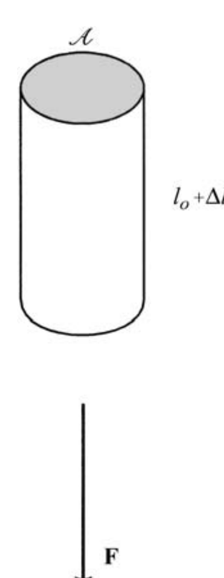

Fig. 1.3.3.1. Bar loaded in pure tension.

references