

1.3. ELASTIC PROPERTIES

Applying relation (1.3.3.2), one gets for the density of strain energy

$$W = \frac{1}{2} T_{ij} S_{ij} = \frac{1}{2} c_{ijkl} S_{ij} S_{kl}. \quad (1.3.3.10)$$

1.3.3.4. Particular elastic constants

1.3.3.4.1. Volume compressibility

Let us apply a hydrostatic pressure (Section 1.3.2.5.2). The medium undergoes a relative variation of volume $\Delta V/V = S_1 + S_2 + S_3$ (the cubic dilatation, Section 1.3.1.3.2). If one replaces in (1.3.3.8) the stress distribution by a hydrostatic pressure, one obtains for the components of the strain tensor

$$\begin{aligned} S_1 &= -p(s_{11} + s_{12} + s_{13}) \\ S_2 &= -p(s_{12} + s_{22} + s_{23}) \\ S_3 &= -p(s_{13} + s_{23} + s_{33}). \end{aligned}$$

From this, we deduce the volume compressibility, χ , which is the inverse of the bulk modulus, κ :

$$\chi = \kappa^{-1} = -\frac{1}{p} \frac{\delta V}{V} = s_{11} + s_{22} + s_{33} + 2(s_{12} + s_{23} + s_{13}). \quad (1.3.3.11)$$

This expression reduces for a cubic or isotropic medium to

$$\chi = \kappa^{-1} = 3(s_{11} + 2s_{12}). \quad (1.3.3.12)$$

1.3.3.4.2. Linear compressibility

Under the action of a hydrostatic pressure, each vector assumes a different elongation. This elongation is given by equation (1.3.1.6):

$$\frac{u_r}{r} = \frac{S_{ij} x_i x_j}{r^2} = S_{ij} \alpha_i \alpha_j = s_{ijkl} T_{kl} \alpha_i \alpha_j,$$

where the α_i 's are the direction cosines of \mathbf{r} . The coefficient of linear compressibility is, by definition, $(-1/p)(u_r/r)$. Replacing T_{kl} by its value $-p\delta_{kl}$, we obtain for the coefficient of linear compressibility

$$-\frac{1}{p} \frac{u_r}{r} = s_{ijkl} \alpha_i \alpha_j.$$

In the case of a cubic or isotropic medium, this expression reduces to

$$-\frac{1}{p} \frac{u_r}{r} = s_{11} + 2s_{12}.$$

The coefficient of linear compressibility is then equal to one third of the coefficient of volume compressibility. We note that the quadric of elongations is a sphere.

1.3.3.4.3. Young's modulus, Poisson's ratio

If the applied stress reduces to a uniaxial stress, $T_{11} = T$, the strain tensor is of the form

$$S_\alpha = s_{1\alpha} T.$$

In particular,

$$S_1 = s_{11} T; \quad S_2 = s_{12} T.$$

We deduce from this that Young's modulus (equation 1.3.3.1) is

$$E = 1/s_{11}. \quad (1.3.3.13)$$

The elongation of a bar under the action of a uniaxial stress is characterized by S_1 and the diminution of the cross section is characterized by S_2 and S_3 . For a cubic material, the relative diminution of the diameter is

$$S_2 = S_3 = s_{12} T.$$

One deduces from this that s_{12} is necessarily of opposite sign to s_{11} and one calls the ratio $\nu = -s_{12}/s_{11}$ Poisson's ratio.

Putting this value into expression (1.3.3.12) for the coefficient of compressibility in cubic or isotropic materials gives

$$\kappa^{-1} = 3s_{11}(1 - 2\nu). \quad (1.3.3.14)$$

As the coefficient of compressibility, by definition, is always positive, we have

$$0 < \nu < 0.5.$$

In practice, Poisson's ratio is always close to 0.3. It is a dimensionless number. The quantity $s_{44}/2(s_{11} - s_{12})$ represents the departure from isotropy of the material and is the anisotropy factor. It is to be noted that cubic materials are not isotropic for elastic properties. Table 1.3.3.2 gives the values of s_{11} , s_{12} , s_{44} , ν and $s_{44}/2(s_{11} - s_{12})$ for a few cubic materials.

Table 1.3.3.2. Elastic compliances of some cubic materials in (GPa)⁻¹ (after Every & McCurdy, 1992)

Material	s_{11}	s_{12}	s_{44}	ν	$s_{44}/2(s_{11} - s_{12})$
Ag	22.9	-9.8	22.1	0.428	0.338
Al	16.0	-5.8	35.3	0.362	0.810
C (diamond)	1.12	-0.14	1.83	0.125	0.726
Cu	15.0	-6.3	13.3	0.42	0.312
Fe	7.67	-2.83	8.57	0.369	0.408
Ge	9.73	-2.64	14.9	0.271	0.602
Mo	2.63	-0.74	9.20	0.273	1.304
Ni	7.67	-2.93	8.23	0.382	0.388
Pb	93.7	-43.04	68.0	0.459	0.249
Si	7.73	-2.15	12.20	0.279	0.636
W	2.45	-0.69	6.24	0.281	0.995
LiF	11.6	-3.35	15.8	0.289	0.528
MgO	4.01	-0.96	6.47	0.239	0.651
NaCl	22.9	-4.8	78.3	0.133	1.452
GaAs	11.75	-3.66	16.8	0.311	0.545
ZnS	19.7	-7.6	22.6	0.386	0.414
BaTiO ₃	8.33	-2.68	9.24	0.322	0.420
Adamantane	240	-79	295	0.329	0.462
Almandine (Fe ₃ Al ₂ Si ₃ O ₁₂)	4.036	-1.093	10.77	0.271	1.050
Spinel (MgAl ₂ O ₄)	5.80	-2.05	6.49	0.353	0.413