

1.3. ELASTIC PROPERTIES

of $[c_{1122} + c_{1221}]$ on the one hand and from that of c_{1221} on the other hand.

1.3.4.4. Polarization of the elastic waves

The Christoffel determinant has three roots and the Christoffel matrix, being Hermitian with real coefficients, has three real eigenvalues and three orthogonal eigenvectors. The wavevector \mathbf{q} , therefore, encompasses three waves with vibration vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ which are perpendicular to one another. In the general case, there is no particular angular relationship between the vibration vectors (or polarization vectors). However, if the latter are parallel to certain symmetry directions in the crystal, one of the vibration vectors is along this direction. The corresponding wave is called longitudinal. The two other waves have their polarization direction perpendicular to the wavevector and are thus transverse. If one of the polarization vectors is almost parallel to the wavevector, which often happens, then one speaks of the vibration as being quasi-longitudinal.

1.3.4.5. Relation between velocity of propagation and elastic stiffnesses

We shall limit ourselves to cubic, hexagonal and tetragonal crystals and consider particular cases.

1.3.4.5.1. Cubic crystals

(i) *The wavevector is parallel to [100].* The Christoffel determinant reduces to

$$\begin{pmatrix} c_{11} - \rho v^2 & 0 & 0 \\ 0 & c_{44} - \rho v^2 & 0 \\ 0 & 0 & c_{44} - \rho v^2 \end{pmatrix} = 0.$$

The three solutions are given in Table 1.3.4.1. These results are valid for a wave propagating in any direction in an isotropic medium.

(ii) *The wavevector is parallel to [110].* The direction cosines of the wavevector are $1/\sqrt{2}, 1/\sqrt{2}, 0$. The Christoffel determinant assumes the form

$$\begin{pmatrix} \frac{1}{2}(c_{11} + c_{44}) - \rho v^2 & \frac{1}{2}(c_{12} + c_{44}) & 0 \\ \frac{1}{2}(c_{12} + c_{44}) & \frac{1}{2}(c_{11} + c_{44}) - \rho v^2 & 0 \\ 0 & 0 & c_{44} - \rho v^2 \end{pmatrix} = 0.$$

The three solutions are given in Table 1.3.4.2.

(iii) *The wavevector is parallel to [111].* The Christoffel determinant assumes the form

$$\begin{pmatrix} c_{11} + 2c_{44} - \rho v^2 & c_{12} + c_{44} & c_{12} + c_{44} \\ c_{12} + c_{44} & c_{11} + 2c_{44} - \rho v^2 & c_{12} + c_{44} \\ c_{12} + c_{44} & c_{12} + c_{44} & c_{44} - \rho v^2 \end{pmatrix} = 0.$$

The solutions are given in Table 1.3.4.3.

1.3.4.5.2. Hexagonal crystals

In hexagonal crystals, there are five independent elastic stiffnesses, $c_{11}, c_{33}, c_{12}, c_{13}, c_{44}$ and $c_{66} = (c_{11} - c_{12})/2$ (Section 1.1.4.10.4).

(i) *The wavevector is parallel to [001].* The Christoffel determinant reduces to

$$\begin{pmatrix} c_{44} - \rho v^2 & 0 & 0 \\ 0 & c_{44} - \rho v^2 & 0 \\ 0 & 0 & c_{33} - \rho v^2 \end{pmatrix} = 0.$$

The solutions are given in Table 1.3.4.4.

(ii) *The wavevector is parallel to [100].* The Christoffel determinant readily reduces to

$$\begin{pmatrix} c_{11} - \rho v^2 & 0 & 0 \\ 0 & c_{66} - \rho v^2 & 0 \\ 0 & 0 & c_{44} - \rho v^2 \end{pmatrix} = 0.$$

The three solutions are given in Table 1.3.4.5.

1.3.4.5.3. Tetragonal crystals (classes 4mm, $\bar{4}2m, 4/mmm$)

In tetragonal crystals, there are six independent elastic stiffnesses, $c_{11}, c_{33}, c_{12}, c_{13}, c_{44}$ and c_{66} (Section 1.1.4.10.4).

(i) *The wavevector is parallel to [001].* The Christoffel determinant reduces to

Table 1.3.4.1. Velocity of propagation when the wavevector is parallel to [100] (cubic crystals)

Velocity of propagation	Polarization vector	Nature of the wave
$v_{\parallel} = \sqrt{c_{11}/\rho}$	[100]	Longitudinal
$v_{\perp} = \sqrt{c_{44}/\rho}$	[010]	Transverse
$v_{\perp} = \sqrt{c_{44}/\rho}$	Any vector normal to [100]	Transverse

Table 1.3.4.2. Velocity of propagation when the wavevector is parallel to [110] (cubic crystals)

Velocity of propagation	Polarization vector	Nature of the wave
$v_{\parallel} = \sqrt{[c_{44} + \frac{1}{2}(c_{11} + c_{12})]/\rho}$	[110]	Longitudinal
$v_{\perp} = \sqrt{\frac{1}{2}(c_{11} - c_{12})/\rho}$	[$\bar{1}10$]	Transverse
$v_{\perp} = \sqrt{c_{44}/\rho}$	[001]	Transverse

Table 1.3.4.3. Velocity of propagation when the wavevector is parallel to [111] (cubic crystals)

Velocity of propagation	Polarization vector	Nature of the wave
$v_{\parallel} = \sqrt{(c_{11} + 2c_{12} + 4c_{44})/3\rho}$	[111]	Longitudinal
$v_{\perp} = \sqrt{(c_{11} - c_{12} + c_{44})/3\rho}$	Any vector normal to [111]	Transverse

Table 1.3.4.4. Velocity of propagation when the wavevector is parallel to [001] (hexagonal crystals)

Velocity of propagation	Polarization vector	Nature of the wave
$v_{\parallel} = \sqrt{c_{33}/\rho}$	[100]	Longitudinal
$v_{\perp} = \sqrt{c_{44}/\rho}$	Any vector normal to [001]	Transverse

Table 1.3.4.5. Velocity of propagation when the wavevector is parallel to [100] (hexagonal crystals)

Velocity of propagation	Polarization vector	Nature of the wave
$v_{\parallel} = \sqrt{c_{11}/\rho}$	[100]	Longitudinal
$v_{\perp} = \sqrt{c_{66}/\rho}$	[010]	Transverse
$v_{\perp} = \sqrt{c_{44}/\rho}$	[001]	Transverse

Table 1.3.4.6. Velocity of propagation when the wavevector is parallel to [001] (tetragonal crystals)

Velocity of propagation	Polarization vector	Nature of the wave
$v_{\parallel} = \sqrt{c_{33}/\rho}$	[100]	Longitudinal
$v_{\perp} = \sqrt{c_{44}/\rho}$	[010]	Transverse
$v_{\perp} = \sqrt{c_{44}/\rho}$	[001]	Transverse

Table 1.3.4.7. Velocity of propagation when the wavevector is parallel to [100] (tetragonal crystals)

Velocity of propagation	Polarization vector	Nature of the wave
$v_{\parallel} = \sqrt{c_{11}/\rho}$	[100]	Longitudinal
$v_{\perp} = \sqrt{c_{66}/\rho}$	[010]	Transverse
$v_{\perp} = \sqrt{c_{44}/\rho}$	[001]	Transverse

1. TENSORIAL ASPECTS OF PHYSICAL PROPERTIES

$$\begin{pmatrix} c_{44} - \rho v^2 & 0 & 0 \\ 0 & c_{44} - \rho v^2 & 0 \\ 0 & 0 & c_{33} - \rho v^2 \end{pmatrix} = 0.$$

The three solutions are given in Table 1.3.4.6.

(ii) *The wavevector is parallel to [100]*. The Christoffel determinant reduces to

$$\begin{pmatrix} c_{11} - \rho v^2 & 0 & 0 \\ 0 & c_{66} - \rho v^2 & 0 \\ 0 & 0 & c_{44} - \rho v^2 \end{pmatrix} = 0.$$

The three solutions are given in Table 1.3.4.7.

1.3.4.6. Experimental determination of elastic constants

1.3.4.6.1. Introduction

As mentioned in Section 1.3.4.1, the elastic constants of a material can be obtained by the elastic response of the material to particular static forces; however, such measurements are not precise and the most often used approach nowadays consists of determining the velocity of ultrasonic waves propagating along different directions of the crystal and calculating the elastic constants from the Christoffel determinants (1.3.4.8). The experimental values are often accurate enough to justify the distinction between static and dynamic values of the elastic constants and between phase and group velocities, and the careful consideration of the frequency range of the experiments.

(i) *Static and dynamic elastic constants*. When one measures the elastic response of a material to external static forces, work is done and heat is produced. In general, the external forces are applied slowly (quasi-elastic processes) and the solid body remains in thermal equilibrium with its surroundings, which can be considered as a heat reservoir. In this case, the measured elastic constants are isothermal elastic constants as defined in Section 1.1.5.1. On the contrary, when ultrasonic waves propagate in a solid body, the compressed regions are slightly hotter than the expanded regions; the associated temperature gradients give rise to irreversible processes of thermal conduction. However, in the megahertz or higher-frequency ranges, the processes of heat exchange between different regions or between the solid and its surroundings are slow compared with the period of the ultrasonic wave and hence can be considered as isentropic. In this case, the measured velocities are isentropic and the elastic stiffnesses deduced from them are the adiabatic elastic stiffnesses as defined in Section 1.1.5.4. The differences between isothermal and adiabatic elastic stiffnesses can be calculated from equation (1.1.5.2):

$$\begin{aligned} \delta T_{ij} &= (c_{ijkl})^\Theta \delta S_{kl} - \lambda_{ij} \delta \Theta \\ \delta \sigma &= \lambda_{kl} \delta S_{kl} + \frac{\rho C^S}{\Theta} \delta \Theta. \end{aligned}$$

By combining these two equations, it is possible to obtain relations between the isothermal elastic stiffnesses, $(c_{ijkl})^\Theta$, and the adiabatic elastic stiffnesses, $(c_{ijkl})^\sigma$:

$$(c_{ijkl})^\sigma = (c_{ijkl})^\Theta + \frac{\lambda_{ij} \lambda_{kl}}{\rho C^S},$$

where c^S is the specific heat at constant strain.

This relation shows that adiabatic elastic stiffnesses are larger than isothermal elastic stiffnesses, at least for the terms of the main diagonal of the elastic matrix. In general, the differences between isothermal and adiabatic elastic stiffnesses are less than 1%. An exception to this statement concerns the temperature region near the critical temperature T_c where a phase transition occurs.

(ii) *Frequency dependence of the elastic constants*. Dynamic measurements of the elastic constants can be performed at a wide range of ultrasonic frequencies. Currently used techniques for investigating a wide range of frequencies are electronic pulse-echo techniques in the megahertz frequency range and Brillouin scattering in the 10 GHz frequency range. To evaluate the possible differences between dynamic elastic constants determined in different frequency ranges, high-accuracy measurements of the *absolute value* of the elastic constants are required. From optical resonance techniques in the megahertz frequency range and Brillouin scattering in the gigahertz frequency range, it has been shown that the results are the same within 0.1 to 0.2% in the case of simple and stable crystals such as sodium chloride when the ultrasonic waves can be considered as a 'passive probe' (Michard *et al.*, 1971). Larger differences are observed when ultrasonic waves interact with the medium, when they favour or impede structural changes of the material or when they are absorbed for a particular frequency.

(iii) *Phase velocity and group velocity*. When the velocity of ultrasonic waves is frequency dependent (dispersive medium), one has to distinguish phase velocity and group velocity. If a resonance technique is used, the resonance frequencies correspond to phase-matching conditions and the calculation of velocity from the resonance frequencies leads to '*phase velocity*'. If a pulse-echo technique is used, the transit time of an elastic pulse is measured and the interpretation of the measurement is more complex. In this case, Brillouin (1932) has shown that the head and the tail of the pulse travel with the group velocity. Differences of a few per cent can be observed between phase and group velocities in dispersive media.

1.3.4.6.2. Resonance technique

The use of the resonance technique is a well established approach for determining the velocity of sound in a gas by observing nodes and antinodes of a system of standing waves produced in the so-called *Kund tube*. In the case of transparent solids, optical means allow us to visualize the standing waves and to measure the wavelength directly (Zarembowitch, 1965). An easier procedure can be used: let us consider a transparent crystal in the shape of a parallelepiped (Fig. 1.3.4.1). A piezoelectric transducer is glued to the crystal and excited at varying frequencies. If the bonding between the transducer and the crystal is loose enough, the crystal can be considered as free from stress and the sequence of its resonance frequencies is given by

$$v = nV/2l,$$

where n is an integer, V the phase velocity of the wave in the direction orthogonal to the parallel faces and l the distance between these faces.

The looseness of the bonding can be checked by the regularity of the arithmetic ratio, $V/2l$. On account of the elasto-optic coupling, a phase grating is associated with the elastic standing-wave system and a light beam can be diffracted by this grating. The intensity of the diffraction pattern is maximum when resonance occurs. A large number of resonance frequencies can be detected, usually more than 100, sometimes 1000 for non-attenuating materials. Consequently, in favourable cases the absolute value of the ultrasonic velocity can be determined with an uncertainty less than 10^{-3} .

1.3.4.6.3. Pulse-echo techniques

Pulse-echo techniques are valid for transparent and opaque materials. They are currently used for measuring ultrasonic velocities in solids and can be used in very simple as well as in sophisticated versions according to the required precision (McSkimmin, 1964). In the simplest version (Fig. 1.3.4.2), an