$$S_{11}=\tfrac{1}{2}\left(\alpha_{11}^2+\alpha_{21}^2+\alpha_{31}^2\right)-1$$
$$=\frac{\partial u_1}{\partial X_1}+\tfrac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_1}\right)^2+\left(\frac{\partial u_2}{\partial X_1}\right)^2+\left(\frac{\partial u_3}{\partial X_1}\right)^2\right]$$
$$S_{12}=S_{21}=\tfrac{1}{2}\frac{\partial u_2}{\partial X_1}$$
$$S_{13}=S_{31}=\tfrac{1}{2}\frac{\partial u_3}{\partial X_1}$$

and the strain invariants reduce to

$$I_1=S_{11};\quad I_2=-(S_{12}S_{21}+S_{13}S_{31});\quad I_3=0.$$

1.3.7.3.1. *Isotropic media*

In this case, the strain-energy density becomes

$$\Phi=\tfrac{1}{2}(\lambda+2\mu)(S_{11})^2+2\mu(S_{12}S_{21}+S_{13}S_{31})+\tfrac{1}{3}(l+2m)(S_{11})^3+2mS_{11}(S_{12}S_{21}+S_{13}S_{31}). \qquad (1.3.7.6)$$

Differentiating (1.3.7.6) with respect to the strains, we get

$$\frac{\partial\Phi}{\partial S_{11}}=(\lambda+2\mu)S_{11}+(l+2m)(S_{11})^2+2m(S_{12}S_{21}+S_{13}S_{31})$$
$$\frac{\partial\Phi}{\partial S_{12}}=2\mu S_{21}+2mS_{11}S_{21}$$
$$\frac{\partial\Phi}{\partial S_{13}}=2\mu S_{31}+2mS_{11}S_{31}$$
$$\frac{\partial\Phi}{\partial S_{21}}=2\mu S_{12}+2mS_{11}S_{12}$$
$$\frac{\partial\Phi}{\partial S_{31}}=2\mu S_{13}+2mS_{11}S_{13}.$$

All the other $\partial\Phi/\partial S_{ij}=0$.

From (1.3.7.5), we derive the stress components:

$$T_{11}=\alpha_{1k}\frac{\partial\Phi}{\partial S_{1k}};\quad T_{12}=\alpha_{2k}\frac{\partial\Phi}{\partial S_{1k}};\quad T_{13}=\alpha_{3k}\frac{\partial\Phi}{\partial S_{1k}};$$
$$T_{21}=\alpha_{1k}\frac{\partial\Phi}{\partial S_{2k}};\quad T_{22}=\alpha_{2k}\frac{\partial\Phi}{\partial S_{2k}};\quad T_{23}=\alpha_{3k}\frac{\partial\Phi}{\partial S_{2k}};$$
$$T_{31}=\alpha_{1k}\frac{\partial\Phi}{\partial S_{3k}};\quad T_{32}=\alpha_{2k}\frac{\partial\Phi}{\partial S_{3k}};\quad T_{33}=\alpha_{3k}\frac{\partial\Phi}{\partial S_{3k}}.$$

Note that this tensor is not symmetric.

For the particular problem discussed here, the three components of the equation of motion are

$$\rho u_1''=\mathrm{d}T_{11}/\mathrm{d}X_1,$$
$$\rho u_2''=\mathrm{d}T_{21}/\mathrm{d}X_1,$$
$$\rho u_3''=\mathrm{d}T_{31}/\mathrm{d}X_1.$$

If we retain only terms up to the quadratic order in the displacement gradients, we obtain the following equations of motion:

$$\rho u_1''=(\lambda+2\mu)\frac{\partial^2u_1}{\partial X_1^2}+[3(\lambda+2\mu)+2(l+2m)]\frac{\partial u_1}{\partial X_1}\frac{\partial^2u_1}{\partial X_1^2}+(\lambda+2\mu+m)\left[\frac{\partial u_2}{\partial X_1}\frac{\partial^2u_2}{\partial X_1^2}+\frac{\partial u_3}{\partial X_1}\frac{\partial^2u_3}{\partial X_1^2}\right]$$
$$\rho u_2''=\mu\frac{\partial^2u_2}{\partial X_1^2}+(\lambda+2\mu+m)\left[\frac{\partial u_1}{\partial X_1}\frac{\partial^2u_2}{\partial X_1^2}+\frac{\partial u_2}{\partial X_1}\frac{\partial^2u_1}{\partial X_1^2}\right]$$
$$\rho u_3''=\mu\frac{\partial^2u_3}{\partial X_1^2}+(\lambda+2\mu+m)\left[\frac{\partial u_1}{\partial X_1}\frac{\partial^2u_3}{\partial X_1^2}+\frac{\partial u_3}{\partial X_1}\frac{\partial^2u_1}{\partial X_1^2}\right]. \qquad (1.3.7.7)$$

1.3.7.3.2. *Cubic media (most symmetrical groups)*

In this case, the strain-energy density becomes

$$\Phi=\tfrac{1}{2}c_{11}(S_{11})^2+c_{44}\left[(S_{12})^2+(S_{21})^2+(S_{31})^2+(S_{13})^2\right]+c_{111}(S_{11})^3+\tfrac{1}{2}c_{166}S_{11}\left[(S_{12})^2+(S_{21})^2+(S_{31})^2+(S_{13})^2\right]. \qquad (1.3.7.8)$$

Differentiating (1.3.7.8) with respect to the strain, one obtains

$$\frac{\partial\Phi}{S_{11}}=c_{11}S_{11}+3c_{111}(S_{11})^2+\tfrac{1}{2}c_{166}[(S_{12})^2+(S_{21})^2+(S_{31})^2+(S_{13})^2)]$$
$$\frac{\partial\Phi}{S_{21}}=2c_{44}S_{21}+c_{166}S_{11}S_{21}$$
$$\frac{\partial\Phi}{S_{31}}=2c_{44}S_{31}+c_{166}S_{11}S_{31}.$$

All other $\partial\Phi/S_{ij}=0$. From (1.3.7.5), we derive the stress components:

$$T_{11}=\alpha_{1k}\frac{\partial\Phi}{\partial S_{1k}}$$
$$T_{21}=\alpha_{1k}\frac{\partial\Phi}{\partial S_{2k}}$$
$$T_{31}=\alpha_{1k}\frac{\partial\Phi}{\partial S_{3k}}.$$

In this particular case, the three components of the equation of motion are

$$\rho u_1''=\mathrm{d}T_{11}/\mathrm{d}X_1$$
$$\rho u_2''=\mathrm{d}T_{21}/\mathrm{d}X_1$$
$$\rho u_3''=\mathrm{d}T_{31}/\mathrm{d}X_1.$$

If we retain only terms up to the quadratic order in the displacement gradients, we obtain the following equations of motion:

$$\rho u_1''=c_{11}\frac{\partial^2u_1}{\partial X_1^2}+[3c_{11}+c_{111}]\frac{\partial u_1}{\partial X_1}\frac{\partial^2u_1}{\partial X_1^2}+(c_{11}+c_{166})\left[\frac{\partial u_2}{\partial X_1}\frac{\partial^2u_2}{\partial X_1^2}+\frac{\partial u_3}{\partial X_1}\frac{\partial^2u_3}{\partial X_1^2}\right]$$
$$\rho u_2''=c_{44}\frac{\partial^2u_2}{\partial X_1^2}+(c_{11}+c_{166})\left[\frac{\partial u_1}{\partial X_1}\frac{\partial^2u_2}{\partial X_1^2}+\frac{\partial u_2}{\partial X_1}\frac{\partial^2u_1}{\partial X_1^2}\right]$$
$$\rho u_3''=c_{44}\frac{\partial^2u_3}{\partial X_1^2}+(c_{11}+c_{166})\left[\frac{\partial u_1}{\partial X_1}\frac{\partial^2u_3}{\partial X_1^2}+\frac{\partial u_3}{\partial X_1}\frac{\partial^2u_1}{\partial X_1^2}\right], \qquad (1.3.7.9)$$

which are identical to (1.3.7.7) if we put