International Tables for Crystallography (2013). Vol. D, Section 1.6.6, pp. 175–176.

Thus

$$\tan\theta = \frac{G_1}{\bar{n}(n_o - n_e)} = \frac{g_{11}}{\bar{n}(n_o - n_e)}. \qquad (1.6.5.45)$$

Generally speaking, the ellipticity is extremely small and difficult to measure (Moxon & Renshaw, 1990). In right-handed quartz (right-handed with respect to optical rotation observed along the c axis), $n_o = 1.544$, $n_e = 1.553$, $g_{11} = g_{22} = -5.82 \times 10^{-5}$ and $g_{33} = 12.96 \times 10^{-5}$ measured at $\lambda = 5100$ Å. Since the c axis is also the optic axis, $\delta = 0$ for the (0001) plane, and thus $\kappa = 1$ for this section [equations (1.6.5.44) and (1.6.5.43)]. This value of $\kappa = 1$ means that the two waves are circular (see Section 1.6.5.5), *i.e.* there is no linear birefringence, only a pure rotation. In this direction, the gyration g_{33} means a rotation of $\rho = 29.5^\circ$ mm^{-1} [using equation (1.6.5.27)]. In a direction normal to the optic axis, from equation (1.6.5.40) one finds $\rho = -13.3^\circ$ mm^{-1}. However, in this direction, the crystal appears linearly birefringent with $\delta = 110.88$ mm^{-1} [equation (1.6.5.42)]. Thus the ellipticity $\kappa = -0.00209$, as calculated from equations (1.6.5.44) and (1.6.5.43). In other words, the two waves are very slightly elliptical, and the sense of rotation of the two ellipses is reversed. Because of the change in sign of the gyration coefficients, it is found that at an angle of 56° 10′ down from the optic axis $\kappa = 0$, meaning that waves travelling along this direction show no optical rotation, only linear birefringence.

1.6.6. Linear electro-optic effect

The linear electro-optic effect, given by $P_i^\omega = \varepsilon_o \chi_{ijk} E_j^\omega E_k^0$, is conventionally expressed in terms of the change in dielectric impermeability caused by imposition of a static electric field on the crystal. Thus one may write the linear electro-optic effect as

$$\Delta\eta_{ij} = r_{ijk} E_k^0, \qquad (1.6.6.1)$$

where the coefficients r_{ijk} form the so-called *linear electro-optic tensor*. These have identical symmetry with the piezoelectric tensor and so obey the same rules (see Table 1.6.6.1). Like the piezoelectric tensor, there is a maximum of 18 independent coefficients (triclinic case) (see Section 1.1.4.10.3). However, unlike in piezoelectricity, in using the Voigt contracted notation there are two major differences:

(1) In writing the electro-optic tensor components as r_{ij}, the first suffix refers to the column number and the second suffix is the row number.

(2) There are no factors of 1/2 or 2.

Typical values of linear electro-optic coefficients are around 10^{-12} mV^{-1}.

1.6.6.1. *Primary and secondary effects*

In considering the electro-optic effect, it is necessary to bear in mind that, in addition to the primary effect of changing the refractive index, the applied electric field may also cause a strain in the crystal *via* the converse piezoelectric effect, and this can then change the refractive index, as a secondary effect, through the elasto-optic effect. Both these effects, which are of comparable magnitude in practice, will occur if the crystal is free. However, if the crystal is mechanically clamped, it is not possible to induce any strain, and in this case therefore only the primary electro-optic effect is seen. In practice, the free and clamped behaviour can be investigated by measuring the linear birefringence when applying electric fields of varying frequencies. When the electric field is static or of low frequency, the effect is measured at constant stress, so that both primary and secondary effects are measured together. For electric fields at frequencies above the natural mechanical resonance of the crystal, the strains are very small, and in this case only the primary effect is measured.

1.6.6.2. *Example of $LiNbO_3$*

In order to understand how tensors can be used in calculating the optical changes induced by an applied electric field, it is instructive to take a particular example and work out the change in refractive index for a given electric field. $LiNbO_3$ is the most widely used electro-optic material in industry and so this forms a useful example for calculation purposes. This material crystallizes in point group $3m$, for which the electro-optic tensor has the form (for the effect of symmetry, see Section 1.1.4.10) (with x_1 perpendicular to m)

$$\begin{pmatrix} 0 & -r_{22} & r_{13} \\ 0 & r_{22} & r_{13} \\ 0 & 0 & r_{33} \\ 0 & r_{51} & 0 \\ r_{51} & 0 & 0 \\ -r_{22} & 0 & 0 \end{pmatrix}, \qquad (1.6.6.2)$$

with $r_{13} = 9.6$, $r_{22} = 6.8$, $r_{33} = 30.9$ and $r_{51} = 32.5 \times 10^{-12}$ mV^{-1}, under the normal measuring conditions where the crystal is unclamped.

Calculation using dielectric impermeability tensor. Suppose, for example, a static electric field E_3^0 is imposed along the x_3 axis. One can then write

$$\begin{pmatrix} \Delta\eta_1 \\ \Delta\eta_2 \\ \Delta\eta_3 \\ \Delta\eta_4 \\ \Delta\eta_5 \\ \Delta\eta_6 \end{pmatrix} = \begin{pmatrix} 0 & -r_{22} & r_{13} \\ 0 & r_{22} & r_{13} \\ 0 & 0 & r_{33} \\ 0 & r_{51} & 0 \\ r_{51} & 0 & 0 \\ -r_{22} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ E_3^0 \end{pmatrix} = \begin{pmatrix} r_{13}E_3^0 \\ r_{13}E_3^0 \\ r_{33}E_3^0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (1.6.6.3)$$

Thus

$$\begin{aligned} \Delta\eta_1 &= r_{13}E^0 = \Delta\eta_2 \\ \Delta\eta_3 &= r_{33}E^0 \\ \Delta\eta_4 &= \Delta\eta_5 = \Delta\eta_6 = 0. \end{aligned} \qquad (1.6.6.4)$$

Since the original indicatrix of $LiNbO_3$ before application of the field is uniaxial,

$$\begin{aligned} \eta_1 &= \frac{1}{n_o^2} = \eta_2 \\ \eta_3 &= \frac{1}{n_e^2}, \end{aligned} \qquad (1.6.6.5)$$

and so differentiating, the following are obtained:

$$\begin{aligned} \Delta\eta_1 &= \Delta\eta_2 = -\frac{2}{n_o^3}\Delta n_o \\ \Delta\eta_3 &= -\frac{2}{n_e^3}\Delta n_e. \end{aligned} \qquad (1.6.6.6)$$

Thus, the induced changes in refractive index are given by

$$\begin{aligned} \Delta n_1 &= \Delta n_2 = -\frac{n_o^3}{2} r_{13}E_3^0 \\ \Delta n_3 &= -\frac{n_e^3}{2} r_{33}E_3^0. \end{aligned} \qquad (1.6.6.7)$$

It can be seen from this that the effect is simply to change the refractive indices by deforming the indicatrix, but maintain the uniaxial symmetry of the crystal. Note that if light is now propagated along, say, x_1, the observed linear birefringence is given by

$$(n_e - n_o) - \tfrac{1}{2}(n_e^3 r_{33} - n_o^3 r_{13})E_3^0. \qquad (1.6.6.8)$$

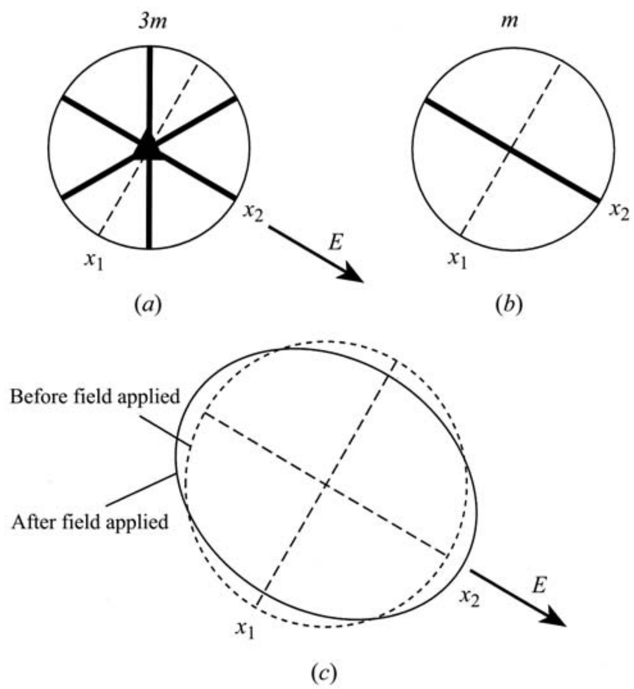

Fig. 1.6.6.1. (*a*) Symmetry elements of point group $3m$. (*b*) Symmetry elements after field applied along x_2. (*c*) Effect on circular section of uniaxial indicatrix.

If, on the other hand, the electric field E_2^0 is applied along x_2, *i.e.* within the mirror plane, one finds

$$\begin{aligned}\Delta\eta_1 &= -r_{22}E_2^0\\ \Delta\eta_2 &= +r_{22}E_2^0\\ \Delta\eta_4 &= +r_{51}E_2^0\\ \Delta\eta_3 &= \Delta\eta_5 = \Delta\eta_6 = 0.\end{aligned} \qquad (1.6.6.9)$$

Diagonalization of the matrix

$$\begin{pmatrix}\Delta\eta_1 & 0 & 0\\ 0 & \Delta\eta_2 & \Delta\eta_4\\ 0 & \Delta\eta_4 & \Delta\eta_3\end{pmatrix} \qquad (1.6.6.10)$$

containing these terms gives three eigenvalue solutions for the changes in dielectric impermeabilities:

$$\begin{aligned}&(1) && -r_{22}E_2^0\\ &(2) && \frac{r_{22}+(r_{22}^2+4r_{51}^2)^{1/2}}{2}E_2^0\\ &(3) && \frac{r_{22}-(r_{22}^2+4r_{51}^2)^{1/2}}{2}E_2^0.\end{aligned} \qquad (1.6.6.11)$$

On calculating the eigenfunctions, it is found that solution (1) lies along x_1, thus representing a change in the value of the indicatrix axis in this direction. Solutions (2) and (3) give the other two axes of the indicatrix: these are different in length, but mutually perpendicular, and lie in the x_2x_3 plane. Thus a biaxial indicatrix is formed with one refractive index fixed along x_1 and the other two in the plane perpendicular. The effect of having the electric field imposed within the mirror plane is thus to remove the threefold axis in point group $3m$ and to form the point subgroup m (Fig. 1.6.6.1).

Relationship between linear electro-optic coefficients r_{ijk} and the susceptibility tensor $\chi_{ijk}^{(2)}$. It is instructive to repeat the above calculation using the normal susceptibility tensor and equation (1.6.3.14). Consider, again, a static electric field along x_3 and light propagating along x_1. As before, the only coefficients that need to be considered with the static field along x_3 are $\chi_{113} = \chi_{223}$ and χ_{333}. Equation (1.6.3.14) can then be written as

$$\begin{pmatrix}\varepsilon_1+\varepsilon_o\chi_{13}E_3^0+n^2 & 0 & 0\\ 0 & \varepsilon_1+\varepsilon_o\chi_{13}E_3^0 & 0\\ 0 & 0 & \varepsilon_3+\varepsilon_o\chi_{33}E_3^0\end{pmatrix}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix} = \begin{pmatrix}n^2 & 0 & 0\\ 0 & n^2 & 0\\ 0 & 0 & n^2\end{pmatrix}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix}, \qquad (1.6.6.12)$$

where for simplicity the Voigt notation has been used. The first line of the matrix equation gives

$$(\varepsilon_1+\varepsilon_o\chi_{13}E_3^0+n^2)E_1 = n^2E_1. \qquad (1.6.6.13)$$

Since only a transverse electric field is relevant for an optical wave (plasma waves are not considered here), it can be assumed that the longitudinal field $E_1 = 0$. The remaining two equations can be solved by forming the determinantal equation

$$\begin{vmatrix}\varepsilon_1+\varepsilon_o\chi_{13}E_3^0-n^2 & 0\\ 0 & \varepsilon_3+\varepsilon_o\chi_{33}E_3^0-n^2\end{vmatrix} = 0, \qquad (1.6.6.14)$$

which leads to the results

$$n_1^2 = \varepsilon_1+\varepsilon_o\chi_{13}E_3^0 \text{ and } n_2^2 = \varepsilon_3+\varepsilon_o\chi_{33}E_3^0. \qquad (1.6.6.15)$$

Thus

$$n_1^2 = n_o^2+\varepsilon_o\chi_{13}E_3^0 \text{ and } n_2^2 = n_e^2+\varepsilon_o\chi_{33}E_3^0, \qquad (1.6.6.16)$$

and so

$$(n_1-n_o)(n_1+n_o) = \varepsilon_o\chi_{13}E_3^0 \text{ and } (n_2-n_e)(n_2+n_e) = \varepsilon_o\chi_{33}E_3^0, \qquad (1.6.6.17)$$

and since $n_1 \simeq n_o$ and $n_2 \simeq n_e$,

$$n_1-n_o = \frac{\varepsilon_o\chi_{13}E_3^0}{2n_o} \text{ and } n_2-n_e = \frac{\varepsilon_o\chi_{33}E_3^0}{2n_e}. \qquad (1.6.6.18)$$

Subtracting these two results, the induced birefringence is found:

$$(n_e-n_o)-\tfrac{1}{2}\left(\frac{\varepsilon_o\chi_{33}}{n_e}-\frac{\varepsilon_o\chi_{13}}{n_o}\right)E_3^0. \qquad (1.6.6.19)$$

Comparing with the equation (1.6.6.8) calculated for the linear electro-optic coefficients,

$$(n_e-n_o)-\tfrac{1}{2}\left(n_e^3r_{33}-n_o^3r_{13}\right)E_3^0, \qquad (1.6.6.20)$$

one finds the following relationships between the linear electro-optic coefficients and the susceptibilities $\chi^{(2)}$:

$$r_{13} = \frac{\varepsilon_o\chi_{13}}{n_o^4} \text{ and } r_{33} = \frac{\varepsilon_o\chi_{33}}{n_e^4}. \qquad (1.6.6.21)$$

1.6.7. The linear photoelastic effect

1.6.7.1. *Introduction*

The linear photoelastic (or piezo-optic) effect (Narasimhamurty, 1981) is given by $P_i^\omega = \varepsilon_o\chi_{ijk\ell}E_j^\omega S_{k\ell}^0$, and, like the electro-optic effect, it can be discussed in terms of the change in dielectric impermeability caused by a static (or low-frequency) field, in this case a stress, applied to the crystal. This can be written in the form

$$\Delta\eta_{ij} = \pi_{ijk\ell}T_{k\ell}^0. \qquad (1.6.7.1)$$