

1.6. CLASSICAL LINEAR CRYSTAL OPTICS

The coefficients π_{ijkl} form a fourth-rank tensor known as the *linear piezo-optic* tensor. Typically, the piezo-optic coefficients are of the order of $10^{-12} \text{ m}^2 \text{ N}^{-1}$. It is, however, more usual to express the effect as an *elasto-optic effect* by making use of the relationship between stress and strain (see Section 1.3.3.2), thus

$$T_{kl} = c_{klmn} S_{mn}, \quad (1.6.7.2)$$

where the c_{klmn} are the elastic stiffness coefficients. Therefore equation (1.6.7.2) can be rewritten in the form

$$\Delta\eta_{ij} = \pi_{ijkl} c_{klmn} S_{mn} = p_{ijmn} S_{mn} \quad (1.6.7.3)$$

or, in contracted notation,

$$\Delta\eta_i = p_{ij} S_j, \quad (1.6.7.4)$$

where, for convenience, the superscript 0 has been dropped, the elastic strain being considered as essentially static or of low frequency compared with the natural mechanical resonances of the crystal. The p_{ijmn} are coefficients that form the *linear elasto-optic* (or *strain-optic*) tensor (Table 1.6.7.1). Note that these coefficients are dimensionless, and typically of order 10^{-1} , showing that the change to the optical indicatrix is roughly one-tenth of the strain.

The elasto-optic effect can arise in several ways. The most obvious way is through application of an external stress, applied to the surfaces of the crystal. However, strains, and hence changes to the refractive indices, can arise in a crystal through other ways that are less obvious. Thus, it is a common finding that crystals can be twinned, and thus the boundary between twin domains, which corresponds to a mismatch between the crystal structures either side of the domain boundary, will exhibit a strain. Such a crystal, when viewed between crossed polars under a microscope will produce birefringence colours that will highlight the contrast between the domains. This is known as *strain birefringence*. Similarly, when a crystal undergoes a phase transition involving a change in crystal system, a so-called *ferroelastic transition*, there will be a change in strain owing to the difference in unit-cell shapes. Hence there will be a corresponding change in the optical indicatrix. Often the phase transition is one going from a high-temperature optically isotropic section to a low-temperature optically anisotropic section. In this case, the high-temperature section has no internal strain, but the low-temperature phase acquires a strain, which is often called the *spontaneous strain* (by analogy with the term spontaneous polarization in ferroelectrics).

 1.6.7.2. Spontaneous strain in BaTiO_3

As an example of the calculation of the relationship between spontaneous strain and linear birefringence, consider the high-temperature phase transition of the well known perovskite BaTiO_3 . This substance undergoes a transition at around 403 K on cooling from its high-temperature $Pm\bar{3}m$ phase to the room-temperature $P4mm$ phase. The $P4mm$ phase is both ferroelectric and ferroelastic. In this tetragonal phase, there is a small distortion of the unit cell along $[001]$ and a contraction along $\langle 100 \rangle$ compared with the unit cell of the high-temperature cubic phase, and so the room-temperature phase can be expected to have a uniaxial optical indicatrix.

The elasto-optic tensor for the $m\bar{3}m$ phase is (Table 1.6.7.1)

$$\begin{pmatrix} p_{11} & p_{12} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{11} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{12} & p_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & p_{44} \end{pmatrix}. \quad (1.6.7.5)$$

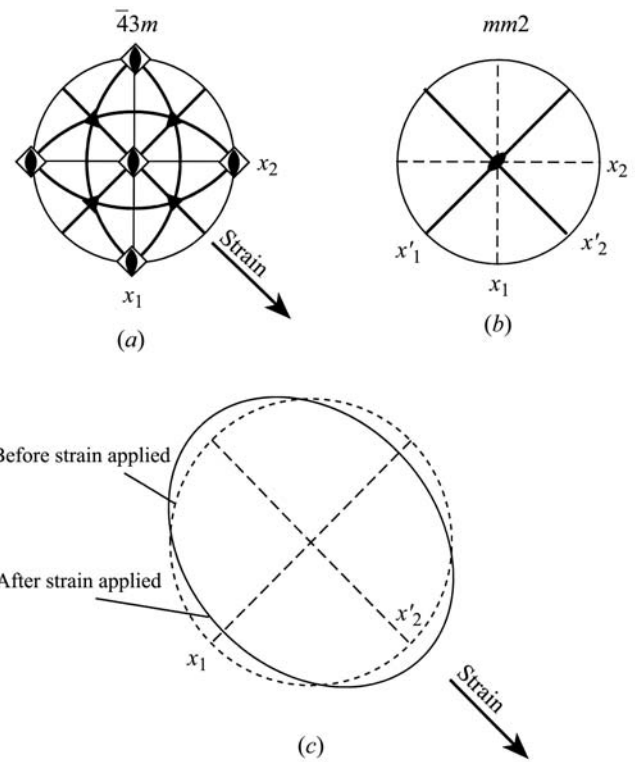


Fig. 1.6.7.1. (a) Symmetry elements of point group $\bar{4}3m$. (b) Symmetry elements after strain applied along $[110]$. (c) Effect on spherical indicatrix.

Consider the low-temperature tetragonal phase to arise as a small distortion of this cubic phase, with a spontaneous strain S_3^s given by the lattice parameters of the tetragonal phase:

$$S_3^s = [(c - a)/a]. \quad (1.6.7.6)$$

Therefore, the equations (1.6.7.4) for the dielectric impermeability in terms of the spontaneous strain component are given in matrix form as

$$\begin{pmatrix} \Delta\eta_1 \\ \Delta\eta_2 \\ \Delta\eta_3 \\ \Delta\eta_4 \\ \Delta\eta_5 \\ \Delta\eta_6 \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{11} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{12} & p_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & p_{44} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ S_3^s \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} p_{12} S_3^s \\ p_{12} S_3^s \\ p_{11} S_3^s \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (1.6.7.7)$$

so that

$$\begin{aligned} \Delta\eta_1 &= \Delta\eta_2 = p_{12} S_3^s \\ \Delta\eta_3 &= p_{11} S_3^s \\ \Delta\eta_4 &= \Delta\eta_5 = \Delta\eta_6 = 0. \end{aligned} \quad (1.6.7.8)$$

By analogy with equations (1.6.6.5) and (1.6.6.6), the induced changes in refractive index are then

1. TENSORIAL ASPECTS OF PHYSICAL PROPERTIES

Table 1.6.7.1. Symmetry constraints on the linear elasto-optic (strain-optic) tensor p_{ij} (contracted notation) (see Section 1.1.4.10.6)

Triclinic	Orthorhombic	Tetragonal	Trigonal
Point group 1	Point groups 222, $mm2$, mmm	Point groups 4, $\bar{4}$, $4/m$	Point groups 3, $\bar{3}$
$\begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} & p_{46} \\ p_{51} & p_{52} & p_{53} & p_{54} & p_{55} & p_{56} \\ p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & p_{66} \end{pmatrix}$	$\begin{pmatrix} p_{11} & p_{12} & p_{13} & 0 & 0 & 0 \\ p_{21} & p_{22} & p_{23} & 0 & 0 & 0 \\ p_{31} & p_{32} & p_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & p_{66} \end{pmatrix}$	$\begin{pmatrix} p_{11} & p_{12} & p_{13} & 0 & 0 & p_{16} \\ p_{12} & p_{11} & p_{13} & 0 & 0 & -p_{16} \\ p_{31} & p_{31} & p_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{44} & p_{45} & 0 \\ 0 & 0 & 0 & -p_{45} & p_{44} & 0 \\ p_{61} & -p_{61} & 0 & 0 & 0 & p_{66} \end{pmatrix}$	$\begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{12} & p_{11} & p_{13} & -p_{14} & -p_{15} & -p_{16} \\ p_{31} & p_{31} & p_{33} & 0 & 0 & 0 \\ p_{41} & -p_{41} & 0 & p_{44} & p_{45} & -p_{51} \\ p_{51} & -p_{51} & 0 & -p_{45} & p_{44} & p_{41} \\ -p_{16} & p_{16} & 0 & -p_{15} & p_{14} & \frac{1}{2}(p_{11} - p_{12}) \end{pmatrix}$
Monoclinic	Point groups 2, m , $2/m$ ($2 \parallel x_2$)	Point groups $4mm$, $\bar{4}2m$, 422 , $4/mmm$	Point groups $3m$, $\bar{3}m$, 32
$\begin{pmatrix} p_{11} & p_{12} & p_{13} & 0 & p_{15} & 0 \\ p_{21} & p_{22} & p_{23} & 0 & p_{25} & 0 \\ p_{31} & p_{32} & p_{33} & 0 & p_{35} & 0 \\ 0 & 0 & 0 & p_{44} & 0 & p_{46} \\ p_{51} & p_{52} & p_{53} & 0 & p_{55} & 0 \\ 0 & 0 & 0 & p_{64} & 0 & p_{66} \end{pmatrix}$	$\begin{pmatrix} p_{11} & p_{12} & p_{13} & 0 & 0 & p_{16} \\ p_{21} & p_{22} & p_{23} & 0 & 0 & p_{26} \\ p_{31} & p_{32} & p_{33} & 0 & 0 & p_{36} \\ 0 & 0 & 0 & p_{44} & p_{45} & 0 \\ 0 & 0 & 0 & 0 & p_{54} & p_{55} \\ p_{61} & p_{62} & p_{63} & 0 & 0 & p_{66} \end{pmatrix}$	$\begin{pmatrix} p_{11} & p_{12} & p_{13} & 0 & 0 & 0 \\ p_{12} & p_{11} & p_{13} & 0 & 0 & 0 \\ p_{31} & p_{31} & p_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & p_{66} \end{pmatrix}$	$\begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} & 0 & 0 \\ p_{12} & p_{11} & p_{13} & -p_{14} & 0 & 0 \\ p_{31} & p_{31} & p_{33} & 0 & 0 & 0 \\ p_{41} & -p_{41} & 0 & p_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{44} & p_{41} \\ 0 & 0 & 0 & 0 & p_{14} & \frac{1}{2}(p_{11} - p_{12}) \end{pmatrix}$

Hexagonal	Cubic	Isotropic
Point groups 6, $\bar{6}$, $6/m$	Point groups $m\bar{3}$, 23	
$\begin{pmatrix} p_{11} & p_{12} & p_{13} & 0 & 0 & p_{16} \\ p_{12} & p_{11} & p_{13} & 0 & 0 & -p_{16} \\ p_{31} & p_{31} & p_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{44} & p_{45} & 0 \\ 0 & 0 & 0 & -p_{45} & p_{44} & 0 \\ -p_{16} & p_{16} & 0 & 0 & 0 & \frac{1}{2}(p_{11} - p_{12}) \end{pmatrix}$	$\begin{pmatrix} p_{11} & p_{12} & p_{21} & 0 & 0 & 0 \\ p_{12} & p_{11} & p_{12} & 0 & 0 & 0 \\ p_{21} & p_{21} & p_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & p_{44} \end{pmatrix}$	$\begin{pmatrix} p_{11} & p_{12} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{11} & p_{13} & 0 & 0 & 0 \\ p_{12} & p_{12} & p_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(p_{11} - p_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(p_{11} - p_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(p_{11} - p_{12}) \end{pmatrix}$
Point groups $6mm$, $\bar{6}m2$, 622 , $6/mmm$	Point groups $\bar{4}3m$, 432 , $m\bar{3}m$	
$\begin{pmatrix} p_{11} & p_{12} & p_{13} & 0 & 0 & 0 \\ p_{12} & p_{11} & p_{13} & 0 & 0 & 0 \\ p_{31} & p_{31} & p_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(p_{11} - p_{12}) \end{pmatrix}$	$\begin{pmatrix} p_{11} & p_{12} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{11} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{12} & p_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & p_{44} \end{pmatrix}$	

$$\Delta n_1 = \Delta n_2 = -\frac{n_{\text{cub}}^3}{2} p_{12} S_3^s$$

$$\Delta n_3 = -\frac{n_{\text{cub}}^3}{2} p_{11} S_3^s, \quad (1.6.7.9)$$

where n_{cub} is the refractive index of the cubic phase. Thus the birefringence in the tetragonal phase as seen by light travelling along x_1 is given by

$$\Delta n_3 - \Delta n_2 = -\frac{n_{\text{cub}}^3}{2} (p_{11} - p_{12}) S_3^s. \quad (1.6.7.10)$$

Thus a direct connection is made between the birefringence of the tetragonal phase of BaTiO_3 and its lattice parameters *via* the spontaneous strain. As in the case of the linear electro-optic effect, the calculation can be repeated using equation (1.6.3.14) with the susceptibilities χ_{11} and χ_{12} to yield the relationship

$$p_{11} = \frac{c_{1111} \epsilon_o \chi_{11}}{n_o^4}; \quad p_{12} = \frac{c_{1122} \epsilon_o \chi_{12}}{n_o^4}. \quad (1.6.7.11)$$

1.6.7.3. The acousto-optic effect

The acousto-optic effect (Sapriel, 1976) is really a variant of the elasto-optic effect, in that the strain field is created by the passage of a sound wave through the crystal. If this wave has frequency ω_1 , the resulting polarization in the presence of a light

wave of frequency ω_2 is given by $P_i^o = \chi_{ijkl} E_j^{\omega_2} S_{kl}^{\omega_1}$, where $\omega = \omega_1 \pm \omega_2$. However, since the sound-wave frequency is very small compared with that of the light, to all intents and purposes the change in frequency of the light field can be ignored. The effect then of the sound wave is to produce within an acousto-optic crystal a spatially modulated change in refractive index: a beam of light can then be diffracted by this spatial modulation, the resulting optical diffraction pattern thus changing with the changing sound signal. Acousto-optic materials therefore can be used as transducers for converting sound signals into optical signals for transmission down optical fibres in communications systems. Consider, for instance, a sound wave propagating along the [110] direction in gallium arsenide (GaAs), which crystallizes in point group $43m$. Suppose that this sound wave is longitudinally polarized. With respect to the cube axes, this corresponds to an oscillatory shear strain $S_{12} \sin(\omega t - k\xi)$, where ξ is a distance along the [110] direction (Fig. 1.6.7.1). Then one can write

$$\Delta \eta_{ij} = p_{ij12} S_{12} \sin(\omega t - k\xi) \quad (1.6.7.12)$$

or in contracted notation

$$\Delta \eta_i = p_{i6} S_6 \sin(\omega t - k\xi). \quad (1.6.7.13)$$

From Table 1.6.7.1, it is seen that the change in dielectric impermeability tensor is