

## 1. TENSORIAL ASPECTS OF PHYSICAL PROPERTIES

tensor  $\chi^{(n)}$  with the full  $\mathbf{E}^n$  tensor. More details on tensor algebra can be found in Chapter 1.1 and in Schwartz (1981).

A more compact expression for (1.7.2.1) is

$$\mathbf{P} = \mathbf{P}_0 + \mathbf{P}_1(t) + \mathbf{P}_2(t) + \dots + \mathbf{P}_n(t) + \dots, \quad (1.7.2.2)$$

where  $\mathbf{P}_0$  represents the static polarization and  $\mathbf{P}_n$  represents the  $n$ th order polarization. The properties of the linear and nonlinear responses will be assumed in the following to comply with time invariance and locality. In other words, time displacement of the applied fields will lead to a corresponding time displacement of the induced polarizations and the polarization effects are assumed to occur at the site of the polarizing field with no remote interactions. In the following, we shall refer to the classical formalism and related notations developed in Butcher (1965) and Butcher & Cotter (1990).

Tensorial expressions will be formulated within the Cartesian formalism and subsequent multiple lower index notation. The alternative irreducible tensor representation, as initially implemented in the domain of nonlinear optics by Jerphagnon *et al.* (1978) and more recently revived by Brasselet & Zyss (1998) in the realm of molecular-engineering studies, is particularly advantageous for connecting the nonlinear hyperpolarizabilities of microscopic (*e.g.* molecular) building blocks of molecular materials to the macroscopic (*e.g.* crystalline) susceptibility level. Such considerations fall beyond the scope of the present chapter, which concentrates mainly on the crystalline level, regardless of the microscopic origin of phenomena.

## 1.7.2.1.1. Linear and nonlinear responses

## 1.7.2.1.1.1. Linear response

Let us first consider the first-order linear response in (1.7.2.1) and (1.7.2.2): the most general possible linear relation between  $\mathbf{P}(t)$  and  $\mathbf{E}(t)$  is

$$\mathbf{P}^{(1)}(t) = \varepsilon_o \int_{-\infty}^{+\infty} d\tau T^{(1)}(t, \tau) \cdot \mathbf{E}(\tau), \quad (1.7.2.3)$$

where  $T^{(1)}$  is a rank-two tensor, or in Cartesian index notation

$$P_{\mu}^{(1)}(t) = \varepsilon_o \int_{-\infty}^{+\infty} d\tau T_{\mu\alpha}^{(1)}(t, \tau) E_{\alpha}(\tau). \quad (1.7.2.4)$$

Applying the time-invariance assumption to (1.7.2.4) leads to

$$\begin{aligned} \mathbf{P}^{(1)}(t + t_0) &= \varepsilon_o \int_{-\infty}^{+\infty} d\tau T^{(1)}(t + t_0, \tau) \cdot \mathbf{E}(\tau) \\ &= \varepsilon_o \int_{-\infty}^{+\infty} d\tau T^{(1)}(t, \tau + t_0) \cdot \mathbf{E}(\tau) \\ &= \varepsilon_o \int_{-\infty}^{+\infty} d\tau' T^{(1)}(t, \tau' - t_0) \cdot \mathbf{E}(\tau'), \end{aligned} \quad (1.7.2.5)$$

hence  $T^{(1)}(t + t_0, \tau) = T^{(1)}(t, \tau - t_0)$  or, setting  $t = 0$  and  $t_0 = t$ ,

$$T^{(1)}(t, \tau) = T^{(1)}(0, \tau - t) = R^{(1)}(t - \tau), \quad (1.7.2.6)$$

where  $R^{(1)}$  is a rank-two tensor referred to as the linear polarization response function, which depends only on the time difference  $t - \tau$ . Substitution in (1.7.2.5) leads to

$$\begin{aligned} \mathbf{P}^{(1)}(t) &= \varepsilon_o \int_{-\infty}^{+\infty} d\tau R^{(1)}(t - \tau) \mathbf{E}(\tau) \\ &= \varepsilon_o \int_{-\infty}^{+\infty} d\tau R^{(1)}(\tau) \mathbf{E}(t - \tau). \end{aligned} \quad (1.7.2.7)$$

$R^{(1)}$  can be viewed as the tensorial analogue of the linear impulse function in electric circuit theory. The causality principle imposes that  $R^{(1)}(\tau)$  should vanish for  $\tau < 0$  so that  $\mathbf{P}^{(1)}(t)$  at time  $t$  will

depend only on polarizing field values before  $t$ .  $R^{(1)}$ ,  $\mathbf{P}^{(1)}$  and  $\mathbf{E}$  are real functions of time.

## 1.7.2.1.1.2. Quadratic response

The most general expression for  $\mathbf{P}^{(2)}(t)$  which is quadratic in  $\mathbf{E}(t)$  is

$$\mathbf{P}^{(2)}(t) = \varepsilon_o \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\tau_1 d\tau_2 T^{(2)}(t, \tau_1, \tau_2) \cdot \mathbf{E}(\tau_1) \otimes \mathbf{E}(\tau_2) \quad (1.7.2.8)$$

or in Cartesian notation

$$P_{\mu}^{(2)}(t) = \varepsilon_o \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\tau_1 d\tau_2 T_{\mu\alpha\beta}^{(2)}(t, \tau_1, \tau_2) E_{\alpha}(\tau_1) E_{\beta}(\tau_2). \quad (1.7.2.9)$$

It can easily be proved by decomposition of  $T^{(2)}$  into symmetric and antisymmetric parts and permutation of dummy variables  $(\alpha, \tau_1)$  and  $(\beta, \tau_2)$ , that  $T^{(2)}$  can be reduced to its symmetric part, satisfying

$$T_{\mu\alpha\beta}^{(2)}(t, \tau_1, \tau_2) = T_{\mu\alpha\beta}^{(2)}(t, \tau_2, \tau_1). \quad (1.7.2.10)$$

From time invariance

$$T^{(2)}(t, \tau_1, \tau_2) = R^{(2)}(t - \tau_1, t - \tau_2), \quad (1.7.2.11)$$

$$\mathbf{P}^{(2)}(t) = \varepsilon_o \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\tau_1 d\tau_2 R^{(2)}(t - \tau_1, t - \tau_2) \cdot \mathbf{E}(\tau_1) \otimes \mathbf{E}(\tau_2),$$

$$\mathbf{P}^{(2)}(t) = \varepsilon_o \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\tau_1 d\tau_2 R^{(2)}(\tau_1, \tau_2) \cdot \mathbf{E}(t - \tau_1) \otimes \mathbf{E}(t - \tau_2). \quad (1.7.2.12)$$

Causality demands that  $R^{(2)}(\tau_1, \tau_2)$  cancels for either  $\tau_1$  or  $\tau_2$  negative while  $R^{(2)}$  is real. Intrinsic permutation symmetry implies that  $R_{\mu\alpha\beta}^{(2)}(\tau_1, \tau_2)$  is invariant by interchange of  $(\alpha, \tau_1)$  and  $(\beta, \tau_2)$  pairs.

## 1.7.2.1.1.3. Higher-order response

The  $n$ th order polarization can be expressed in terms of the  $(n + 1)$ -rank tensor  $T^{(n)}(t, \tau_1, \tau_2, \dots, \tau_n)$  as

$$\begin{aligned} \mathbf{P}^{(n)}(t) &= \varepsilon_o \int_{-\infty}^{+\infty} d\tau_1 \int_{-\infty}^{+\infty} d\tau_2 \dots \int_{-\infty}^{+\infty} d\tau_n T^{(n)}(t, \tau_1, \tau_2, \dots, \tau_n) \\ &\quad \cdot \mathbf{E}(\tau_1) \otimes \mathbf{E}(\tau_2) \otimes \dots \otimes \mathbf{E}(\tau_n). \end{aligned} \quad (1.7.2.13)$$

For similar reasons to those previously stated, it is sufficient to consider the symmetric part of  $T^{(n)}$  with respect to the  $n!$  permutations of the  $n$  pairs  $(\alpha_1, \tau_1), (\alpha_2, \tau_2) \dots (\alpha_n, \tau_n)$ . The  $T^{(n)}$  tensor will then exhibit intrinsic permutation symmetry at the  $n$ th order. Time-invariance considerations will then allow the introduction of the  $(n + 1)$ th-rank real tensor  $R^{(n)}$ , which generalizes the previously introduced  $R$  operators:

$$\begin{aligned} \mathbf{P}_{\mu}^{(n)}(t) &= \varepsilon_o \int_{-\infty}^{+\infty} d\tau_1 \int_{-\infty}^{+\infty} d\tau_2 \dots \int_{-\infty}^{+\infty} d\tau_n R_{\mu\alpha_1\alpha_2\dots\alpha_n}^{(n)}(\tau_1, \tau_2, \dots, \tau_n) \\ &\quad \times E_{\alpha_1}(t - \tau_1) E_{\alpha_2}(t - \tau_2) \dots E_{\alpha_n}(t - \tau_n). \end{aligned} \quad (1.7.2.14)$$

$R^{(n)}$  cancels when one of the  $\tau_i$ 's is negative and is invariant under any of the  $n!$  permutations of the  $(\alpha_i, \tau_i)$  pairs.

## 1.7.2.1.2. Linear and nonlinear susceptibilities

Whereas the polarization response has been expressed so far in the time domain, in which causality and time invariance are most naturally expressed, Fourier transformation into the frequency domain permits further simplification of the equations given above and the introduction of the susceptibility tensors according to the following derivation.