International Tables for Crystallography (2013). Vol. D, Section 1.7.3.1.1, pp. 186–188.

lographical frame by the standard conventions given in Chapter 1.6.

1.7.3. Propagation phenomena

1.7.3.1. *Crystalline linear optical properties*

We summarize here the main linear optical properties that govern the nonlinear propagation phenomena. The reader may refer to Chapter 1.6 for the basic equations.

1.7.3.1.1. *Index surface and electric field vectors*

The relations between the different field vectors relative to a propagating electromagnetic wave are obtained from the constitutive relations of the medium and from Maxwell equations.

In the case of a non-magnetic and non-conducting medium, Maxwell equations lead to the following wave propagation equation for the Fourier component at the circular frequency ω defined by (1.7.2.15) and (1.7.2.16) (Butcher & Cotter, 1990):

$$\nabla\mathbf{x}\nabla\mathbf{x}\mathbf{E}(\omega) = (\omega^2/c^2)\mathbf{E}(\omega) + \omega^2\mu_0\mathbf{P}(\omega), \qquad (1.7.3.1)$$

where $\omega = 2\pi c/\lambda$, λ is the wavelength and c is the velocity of light in a vacuum; μ_0 is the free-space permeability, $\mathbf{E}(\omega)$ is the electric field vector and $\mathbf{P}(\omega)$ is the polarization vector.

Table 1.7.2.3. *Nonzero $\chi^{(2)}$ coefficients and equalities between them under the Kleinman symmetry assumption*

Symmetry class	Independent nonzero $\chi^{(2)}$ elements under Kleinman symmetry
Triclinic	
C_1 (1)	xxx, $xyy = yxy = yyx$, $xzz = zxz = zzx$, $xyz = xzy = yxz = yzx = zxy = zyx$, $xxz = xzx = zxx$, $xxy = xyx = yxx$, yyy, $yzz = zyz = zzy$, $yyz = yzy = zyy$, zzz
Monoclinic	
C_2 (2) (twofold axis parallel to z)	$xyz = xzy = yxz = yzx = zxy = zyx$, $xxz = xzx = zxx$, $yyz = yzy = zyy$, zzz
C_s (m) (mirror perpendicular to z)	xxx, $xyy = yxy = yyx$, $xzz = zxz = zzx$, $xxy = xyx = yxx$, yyy, $yzz = zyz = zzy$
Orthorhombic	
C_{2v} ($mm2$) (twofold axis parallel to z)	$xzx = xxz = zxx$, $yyz = yzy = zyy$, zzz
D_2 (222)	$xyz = xzy = yzx = yxz = zxy = zyx$
Tetragonal	
C_4 (4)	$xzx = xxz = zxx = yzy = yyz = zyy$, zzz
S_4 ($\bar{4}$)	$xyz = xzy = yzx = yxz = zxy = zyx$, $xzx = xxz = zxx = -yzy = -yyz = -zyy$
D_4 (422)	All elements are nil
C_{4v} ($4mm$)	$xzx = xxz = zxx = yyz = yzy = zyy$, zzz
D_{2d} ($\bar{4}2m$)	$xyz = xzy = yzx = yxz = zxy = zyx$
Hexagonal	
C_6 (6)	$xzx = xxz = zxx = yyz = yzy = zyy$, zzz
C_{3h} ($\bar{6}$)	$xxx = -xyy = -yxy = -yyx$, $yyy = -yxx = -xyx = -xxy$
D_6 (622)	All elements are nil
C_{6v} ($6mm$)	$xzx = xxz = zxx = yyz = yzy = zyy$, zzz
D_{3h} ($\bar{6}2m$) (mirror perpendicular to x)	$yyy = -yxx = -xxy = -xyx$
Trigonal	
C_3 (3)	$xxx = -xyy = -yyx = -yxy$, $xzx = xxz = zxx = yyz = yzy = zyy$, $yyy = -yxx = -xxy = -xyx$, zzz
D_3 (32)	$xxx = -xyy = -yyx = -yxy$
C_{3v} ($3m$) (mirror perpendicular to x)	$yyy = -yxx = -xxy = -xyx$, $xzx = xxz = zxx = yyz = yzy = zyy$, zzz
Cubic	
T (23), T_d ($\bar{4}3m$)	$xyz = xzy = yzx = yxz = zxy = zyx$
O (432)	All elements are nil

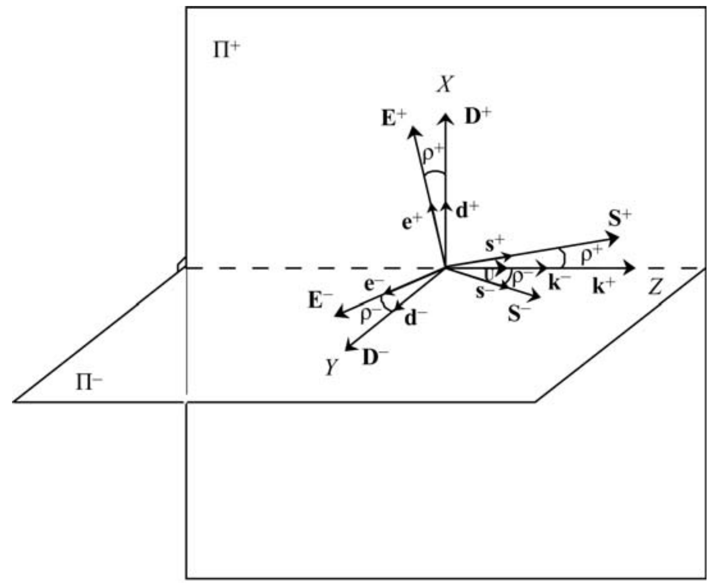

Fig. 1.7.3.1. Field vectors of a plane wave propagating in an anisotropic medium. (X, Y, Z) is the wave frame. Z is along the direction of propagation, X and Y are contained in Π^+ and Π^- respectively, by an arbitrary convention.

In the linear regime, $\mathbf{P}(\omega) = \varepsilon_0\chi^{(1)}(\omega)\mathbf{E}(\omega)$, where ε_0 is the free-space permittivity and $\chi^{(1)}(\omega)$ is the first-order electric susceptibility tensor. Then (1.7.3.1) becomes

$$\nabla\mathbf{x}\nabla\mathbf{x}\mathbf{E}(\omega) = (\omega^2/c^2)\varepsilon(\omega)\mathbf{E}(\omega). \qquad (1.7.3.2)$$

$\varepsilon(\omega) = 1 + \chi^{(1)}(\omega)$ is the dielectric tensor. In the general case, $\chi^{(1)}(\omega)$ is a complex quantity *i.e.* $\chi^{(1)} = \chi^{(1)'} + i\chi^{(1)''}$. For the following, we consider a medium for which the losses are small ($\chi^{(1)'} \gg \chi^{(1)''}$); it is one of the necessary characteristics of an efficient nonlinear medium. In this case, the dielectric tensor is real: $\varepsilon = 1 + \chi^{(1)'}$.

The plane wave is a solution of equation (1.7.3.2):

$$\mathbf{E}(\omega, X, Y, Z) = \mathbf{e}(\omega)E(\omega, X, Y, Z)\exp[\pm ik(\omega)Z]. \qquad (1.7.3.3)$$

(X, Y, Z) is the orthonormal frame linked to the wave, where Z is along the direction of propagation.

We consider a linearly polarized wave so that the unit vector $\mathbf{e}$ of the electric field is real ($\mathbf{e} = \mathbf{e}^*$), contained in the XZ or YZ planes.

$E(\omega, X, Y, Z) = A(\omega, X, Y, Z)\exp[i\Phi(\omega, Z)]$ is the scalar complex amplitude of the electric field where $\Phi(\omega, Z)$ is the phase, and $E^*(-\omega, X, Y, Z) = E(\omega, X, Y, Z)$. In the linear regime, the amplitude of the electric field varies with Z only if there is absorption.

k is the modulus of the wavevector, real in a lossless medium: $+kZ$ corresponds to forward propagation along Z, and $-kZ$ to backward propagation. We consider that the plane wave propagates in an anisotropic medium, so there are two possible wavevectors, $\mathbf{k}^+$ and $\mathbf{k}^-$, for a given direction of propagation of unit vector $\mathbf{u}$:

$$\mathbf{k}^{\pm}(\omega, \theta, \varphi) = (\omega/c)n^{\pm}(\omega, \theta, \varphi)\mathbf{u}(\theta, \varphi). \qquad (1.7.3.4)$$

(θ, φ) are the spherical coordinates of the direction of the unit wavevector $\mathbf{u}$ in the optical frame; (x, y, z) is the optical frame defined in Section 1.7.2.

The spherical coordinates are related to the Cartesian coordinates (u_x, u_y, u_z) by

$$u_x = \cos\varphi\sin\theta \quad u_y = \sin\varphi\sin\theta \quad u_z = \cos\theta. \qquad (1.7.3.5)$$

The refractive indices $n^{\pm}(\omega, \theta, \varphi) = [\varepsilon^{\pm}(\omega, \theta, \varphi)]^{1/2}$, $(n^+ > n^-)$, real in the case of a lossless medium, are the two solutions of the Fresnel equation (Yao & Fahlen, 1984):

$$
\begin{aligned}
n^{\pm} &= \left[\frac{2}{-B \mp (B^2 - 4C)^{1/2}}\right]^{1/2} \\
B &= -u_x^2(b+c) - u_y^2(a+c) - u_z^2(a+b) \\
C &= u_x^2 bc + u_y^2 ac + u_z^2 ab \\
a &= n_x^{-2}(\omega), \quad b = n_y^{-2}(\omega), \quad c = n_z^{-2}(\omega).
\end{aligned}
\tag{1.7.3.6}
$$

$n_x(\omega)$, $n_y(\omega)$ and $n_z(\omega)$ are the principal refractive indices of the index ellipsoid at the circular frequency ω.

Equation (1.7.3.6) describes a double-sheeted three-dimensional surface: for a direction of propagation $\mathbf{u}$ the distances from the origin of the optical frame to the sheets (+) and (−) correspond to the roots n^+ and n^-. This surface is called the index surface or the wavevector surface. The quantity $(n^+ - n^-)$ or $(n^- - n^+)$ is the birefringency. The waves (+) and (−) have the phase velocities c/n^+ and c/n^-, respectively.

Equation (1.7.3.6) and its dispersion in frequency are often used in nonlinear optics, in particular for the calculation of the phase-matching directions which will be defined later. In the regions of transparency of the crystal, the frequency law is well described by a Sellmeier equation, which is the case for normal dispersion where the refractive indices increase with frequency (Hadni, 1967):

$$n^{\pm}(\omega_i) < n^{\pm}(\omega_j) \quad \text{for} \quad \omega_i < \omega_j. \tag{1.7.3.7}$$

If ω_i or ω_j are near an absorption peak, even weak, $n^{\pm}(\omega_i)$ can be greater than $n^{\pm}(\omega_j)$; this is called abnormal dispersion.

The dielectric displacements $\mathbf{D}^{\pm}$, the electric fields $\mathbf{E}^{\pm}$, the energy flux given by the Poynting vector $\mathbf{S}^{\pm} = \mathbf{E}^{\pm} \times \mathbf{H}^{\pm}$ and the collinear wavevectors $\mathbf{k}^{\pm}$ are coplanar and define the orthogonal vibration planes $\Pi^{\pm}$ (Shuvalov, 1981). Because of anisotropy, $\mathbf{k}^{\pm}$ and $\mathbf{S}^{\pm}$, and hence $\mathbf{D}^{\pm}$ and $\mathbf{E}^{\pm}$, are non-collinear in the general case as shown in Fig. 1.7.3.1: the walk-off angles, also termed double-refraction angles, $\rho^{\pm} = \arccos(\mathbf{d}^{\pm} \cdot \mathbf{e}^{\pm}) = \arccos(\mathbf{u} \cdot \mathbf{s}^{\pm})$ are different in the general case; $\mathbf{d}^{\pm}$, $\mathbf{e}^{\pm}$, $\mathbf{u}$ and $\mathbf{s}^{\pm}$ are the unit vectors associated with $\mathbf{D}^{\pm}$, $\mathbf{E}^{\pm}$, $\mathbf{k}^{\pm}$ and $\mathbf{S}^{\pm}$, respectively. We shall see later that the efficiency of a nonlinear interaction is strongly conditioned by $\mathbf{k}$, $\mathbf{E}$ and ρ, which only depend on $\chi^{(1)}(\omega)$, that is to say on the linear optical properties.

The directions $\mathbf{S}^+$ and $\mathbf{S}^-$ are the directions normal to the sheets (+) and (−) of the index surface at the points n^+ and n^-.

For a plane wave, the time-average Poynting vector is (Yariv & Yeh, 2002)

$$
\begin{aligned}
\left\|\mathbf{S}^{\pm}(\omega)\right\| &= \left\|\tfrac{1}{2} Re\left[\mathbf{E}^{\pm}(\omega) \times \mathbf{H}^{\pm *}(\omega)\right]\right\| \\
&= \tfrac{1}{2} \frac{\left\|\mathbf{k}^{\pm}(\omega)\right\|}{\mu_0 \omega} \left\|\mathbf{E}^{\pm}(\omega)\right\|^2 \cos^2 \rho^{\pm}(\omega).
\end{aligned}
\tag{1.7.3.8}
$$

$\|\mathbf{S}^{\pm}\|$ is the energy flow $I = \hbar\omega N^{\pm}$, which is a power per unit area *i.e.* the intensity, where $\hbar\omega$ is the energy of the photon and $N^{\pm}$ are the photons flows. $\rho^{\pm}(\omega)$ is the angle between $\mathbf{S}^{\pm}$ and $\mathbf{u}$; it is detailed later on.

The unit electric field vectors $\mathbf{e}^+$ and $\mathbf{e}^-$ are calculated from the propagation equation projected on the three axes of the optical

Table 1.7.2.4. *Nonzero $\chi^{(3)}$ coefficients and equalities between them in the general case*

Symmetry class	$\chi^{(3)}$ nonzero elements
Triclinic	
C_1 (1), C_i ($\bar{1}$)	All 81 elements are independent and nonzero
Monoclinic	
C_s (*m*), C_2 (2), C_{2h} ($\frac{2}{m}$) (twofold axis parallel to *z*)	*xxxx, xyyy, xyzz, xzyz, xzzy, xxzz, xzxz, xzzx, xxyy, xyxy, xyyx, xxxy, xxyx, xyxx, yxxx, yyyy, yyzz, yzyz, yzzy, yxzz, yzxz, yzzx, yxyy, yyxy, yyyx, yxxy, yxyx, yyxx, zzzz, zyyz, zyzy, zzyy, zxxz, zxzx, zzxx, zxyz, zxzy, zyxz, zzxy, zyzx, zzyx*
Orthorhombic	
C_{2v} (*mm*2), D_2 (222), D_{2h} (*mmm*) (twofold axis parallel to *z*)	*xxxx, xxzz, xzxz, xzzx, xxyy, xyxy, xyyx, yyyy, yyzz, yzyz, yzzy, yxxy, yxyx, yyxx, zzzz, zyyz, zyzy, zzyy, zxxz, zxzx, zzxx*
Tetragonal	
S_4 ($\bar{4}$), C_4 (4), C_{4h} ($\frac{4}{m}$)	*xxxx = yyyy, xyyy = −yxxx, xyzz = −yxzz, xzyz = −yzxz, xzzy = −yzzx, xxzz = yyzz, xzxz = yzyz, xzzx = yzzy, xxyy = yyxx, xyxy = yxyx, xyyx = yxxy, xxxy = −yyyx, xxyx = −yyxy, xyxx = −yxyy, zzzz, zyyz = zxxz, zyzy = zxzx, zzyy = zzxx, zxyz = −zyxz, zxzy = −zyzx, zzxy = −zzyx*
C_{4v} (4*mm*), D_{2d} ($\bar{4}$2*m*), D_4 (422), D_{4h} ($\frac{4}{m}$*mm*)	*xxxx = yyyy, xxzz = yyzz, xzxz = yzyz, xzzx = yzzy, xxyy = yyxx, xyxy = yxyx, xyyx = yxxy, zzzz, zyyz = zxxz, zyzy = zxzx, zzyy = zzxx*
Hexagonal	
C_{3h} ($\bar{6}$), C_6 (6), C_{6h} ($\frac{6}{m}$)	*xxxx = yyyy = xxyy + xyxy + xyyx, xyyy = xxxy + xxyx + xyxx = −yxxx, xyzz = −yxzz, xzyz = −yzxz, xzzy = −yzzx, xxzz = yyzz, xzxz = yzyz, xzzx = yzzy, xxyy = yyxx, xyxy = yxyx, xyyx = yxxy, xxxy = −yyyx, xxyx = −yyxy, xyxx = −yxyy, zzzz, zyyz = zxxz, zyzy = zxzx, zzyy = zzxx, zxyz = −zyxz, zxzy = −zyzx, zzxy = −zzyx*
C_{6v} (6*mm*), D_{3h} ($\bar{6}$2*m*), D_6 (622), D_{6h} ($\frac{6}{m}$*mm*)	*xxxx = yyyy = xxyy + xyxy + xyyx, xxzz = yyzz, xzxz = yzyz, xzzx = yzzy, xxyy = yyxx, xyxy = yxyx, xyyx = yxxy, zzzz, zyyz = zxxz, zyzy = zxzx, zzyy = zzxx*
Trigonal	
C_3 (3), C_{3i} ($\bar{3}$)	*xxxx = yyyy = xxyy + xyxy + xyyx, xyyy = xxxy + xxyx + xyxx = −yxxx, xyzz = −yxzz, xzyz = −yzxz, xzzy = −yzzx, xyyz = yxyz = yyxz = −xxxz, xyzy = yyzx = yxzy = −xxzx, xzyy = yzxy = yzyx = −xzxx, xxzz = yyzz, xzxz = yzyz, xzzx = yzzy, xxyy = yyxx, xyxy = yxyx, xyyx = yxxy, xxxy = −yyyx, xxyx = −yyxy, xyxx = −yxyy, xxyz = xyxz = yxxz = −yyyz, xxzy = xyzx = yxzx = −yyzy, xzxy = xzyx = yzxx = −yzyy, −zxxx = zxyy = zyxy = zyyx, −zyyy = zxxy = zxyx = zyxx, zzzz, zyyz = zxxz, zyzy = zxzx, zzyy = zzxx, zxyz = −zyxz, zxzy = −zyzx, zzxy = −zzyx*
C_{3v} (3*m*), D_3 (32), D_{3d} ($\bar{3}$*m*) (mirror perpendicular to *x*) (twofold axis parallel to *x*)	*xxxx = yyyy = xxyy + xyxy + xyyx, xxzz = yyzz, xzxz = yzyz, xzzx = yzzy, xxyy = yyxx, xyxy = yxyx, xyyx = yxxy, xxyz = xyxz = yxxz = −yyyz, xxzy = xyzx = yxzx = −yyzy, xzxy = xzyx = yzxx = −yzyy, −zyyy = zxxy = zxyx = zyxx, zzzz, zyyz = zxxz, zyzy = zxzx, zzyy = zzxx*
Cubic	
T (23), T_h (*m*3)	*xxxx = yyyy = zzzz, xxzz = yyxx = zzyy, xzxz = yxyx = zyzy, xzzx = yxxy = zyyz, xxyy = yyzz = zzxx, xyxy = yzyz = zxzx, xyyx = yzzy = zxxz*
T_d ($\bar{4}$3*m*), *O* (432), O_h (*m*3*m*)	*xxxx = yyyy = zzzz, xxzz = xxyy = yyzz = yyxx = zzyy = zzxx, xzxz = xyxy = yzyz = yxyx = zyzy = zxzx, xzzx = xyyx = yzzy = yxxy = zyyz = zxxz*

frame. We obtain, for each wave, three equations which relate the three components (e_x, e_y, e_z) to the unit wavevector components (u_x, u_y, u_z) (Shuvalov, 1981):

$$(n^{\pm})^2(e_p^{\pm} - u_p[u_x e_x^{\pm} + u_y e_y^{\pm} + u_z e_z^{\pm}]) = (n_p)^2 e_p^{\pm} \quad (p = x, y \text{ and } z) \tag{1.7.3.9}$$

with $(e_x^{\pm})^2 + (e_y^{\pm})^2 + (e_z^{\pm})^2 = 1$.

The vibration planes $\Pi^{\pm}$ relative to the eigen polarization modes $\mathbf{e}^{\pm}$ are called the neutral vibration planes associated with $\mathbf{u}$: an incident linearly polarized wave with a vibration plane parallel to Π^{+} or Π^{-} is refracted inside the crystal without depolarization, that is to say in a linearly polarized wave, $\mathbf{e}^{+}$ or $\mathbf{e}^{-}$, respectively. For any other incident polarization the wave is refracted in the two waves $\mathbf{e}^{+}$ and $\mathbf{e}^{-}$, which propagate with the difference of phase $(\omega/c)(n^{+} - n^{-})Z$.

The existence of equalities between the principal refractive indices determines the three optical classes: isotropic for the cubic system; uniaxial for the tetragonal, hexagonal and trigonal systems; and generally biaxial for the orthorhombic, monoclinic and triclinic systems [Nye (1957) and Sections 1.1.4.1 and 1.6.3.2].

1.7.3.1.2. *Isotropic class*

The isotropic class corresponds to the equality of the three principal indices: the index surface is a one-sheeted sphere, so $n^{+} = n^{-}$, $\rho^{+} = \rho^{-} = 0$ for all directions of propagation, and any electric field vector direction is allowed as in an amorphous material.

1.7.3.1.3. *Uniaxial class*

The uniaxial class is characterized by the equality of two principal indices, called ordinary indices $(n_x = n_y = n_o)$; the other index is called the extraordinary index $(n_z = n_e)$. Then, according to (1.7.3.6), the index surface has one umbilicus along the z axis, $n^{+}(\theta = 0) = n^{-}(\theta = 0)$, called the optic axis, which is along the fold rotation axis of greatest order of the crystal. The two other principal axes are related to the symmetry elements of the orientation class according to the standard conventions (Nye, 1957). The ordinary sheet is spherical *i.e.* $n_o(\theta, \varphi) = n_o$, so an ordinary wave has no walk-off for any direction of propagation in a uniaxial crystal; the extraordinary sheet is ellipsoidal *i.e.* $n_e(\theta, \varphi) = [(\cos^2\theta)/(n_o^2) + (\sin^2\theta)/(n_e^2)]^{-1/2}$. The sign of the uniaxial class is defined by the sign of the birefringence $n_e - n_o$. Thus, according to these definitions, (n_e, n_o) corresponds to (n^{+}, n^{-}) for the positive class $(n_e > n_o)$ and to (n^{-}, n^{+}) for the negative class $(n_e < n_o)$, as shown in Fig. 1.7.3.2.

The ordinary electric field vector is orthogonal to the optic axis $(e_z^o = 0)$, and also to the extraordinary electric field vector, leading to

$$\mathbf{e}^o(\omega_i, \theta, \varphi) \cdot \mathbf{e}^e(\omega_j, \theta, \varphi) = 0. \tag{1.7.3.10}$$

This relation is satisfied when ω_i and ω_j are equal or different and for any direction of propagation (θ, φ).

According to these results, the coplanarity of the field vectors imposes the condition that the double-refraction angle of the extraordinary wave is in a plane containing the optic axis. Thus, the components of the ordinary and extraordinary unit electric field vectors $\mathbf{e}^o$ and $\mathbf{e}^e$ at the circular frequency ω are

$$e_x^o = -\sin\varphi \quad e_y^o = +\cos\varphi \quad e_z^o = 0 \tag{1.7.3.11}$$

$$\begin{aligned} e_x^e &= -\cos[\theta \pm \rho^{\mp}(\theta, \omega)] \cdot \cos\varphi \\ e_y^e &= -\cos[\theta \pm \rho^{\mp}(\theta, \omega)] \cdot \sin\varphi \\ e_z^e &= \sin[\theta \pm \rho^{\mp}(\theta, \omega)] \end{aligned} \tag{1.7.3.12}$$

Table 1.7.2.5. *Nonzero $\chi^{(3)}$ coefficients and equalities between them under the Kleinman symmetry assumption*

Symmetry class	Independent nonzero elements of $\chi^{(3)}$ under Kleinman symmetry
Triclinic C_1 (1), C_i ($\bar{1}$)	*xxxx*, *xyyy* = *yxyy* = *yyxy* = *yyyx*, *xzzz* = *zxzz* = *zzxz* = *zzzx*, *xyzz* = *xzyz* = *xzzy* = *yxzz* = *yzxz* = *yzzx* = *zxyz* = *zxzy* = *zyxz* = *zyzx* = *zzxy* = *zzyx*, *xyyz* = *xyzy* = *xzyy* = *yxyz* = *yxzy* = *yyxz* = *yyzx* = *yzxy* = *yzyx* = *zxyy* = *zyxy* = *zyyx*, *xxzz* = *xzxz* = *xzzx* = *zxxz* = *zxzx* = *zzxx*, *xxxz* = *xxzx* = *xzxx* = *zxxx*, *xxyy* = *xyxy* = *xyyx* = *yxxy* = *yxyx* = *yyxx*, *xxxy* = *xxyx* = *xyxx* = *yxxx*, *xxyz* = *xxzy* = *xyxz* = *xyzx* = *xzxy* = *xzyx* = *yxxz* = *yxzx* = *yzxx* = *zxxy* = *zxyx* = *zyxx*, *yyyy*, *yzzz* = *zyzz* = *zzyz* = *zzzy*, *yyzz* = *yzyz* = *yzzy* = *zyyz* = *zyzy* = *zzyy*, *yyyz* = *yyzy* = *yzyy* = *zyyy*, *zzzz*
Monoclinic C_s (*m*), C_2 (2), C_{2h} $\left(\frac{2}{m}\right)$ (twofold axis parallel to *z*)	*xxxx*, *xyyy* = *yxyy* = *yyxy* = *yyyx*, *xyzz* = *xzyz* = *xzzy* = *yxzz* = *yzxz* = *yzzx* = *zxyz* = *zxzy* = *zyxz* = *zyzx* = *zzxy* = *zzyx*, *xxzz* = *xzxz* = *xzzx* = *zxxz* = *zxzx* = *zzxx*, *xxyy* = *xyxy* = *xyyx* = *yxxy* = *yxyx* = *yyxx*, *xxxy* = *xxyx* = *xyxx* = *yxxx*, *yyyy*, *yyzz* = *yzyz* = *yzzy* = *zyyz* = *zyzy* = *zzyy*, *zzzz*
Orthorhombic C_{2v} (*mm*2), D_2 (222), D_{2h} (*mmm*) (twofold axis parallel to *z*)	*xxxx*, *xxzz* = *xzxz* = *xzzx* = *zxxz* = *zxzx* = *zzxx*, *xxyy* = *xyxy* = *xyyx* = *yxxy* = *yxyx* = *yyxx*, *yyyy*, *yyzz* = *yzyz* = *yzzy* = *zyyz* = *zyzy* = *zzyy*, *zzzz*
Tetragonal S_4 ($\bar{4}$), C_4 (4), C_{4h} $\left(\frac{4}{m}\right)$	*xxxx* = *yyyy*, *xyyy* = *yxyy* = *yyxy* = *yyyx* = −*xxxy* = −*xxyx* = −*xyxx* = −*yxxx*, *xxzz* = *xzxz* = *xzzx* = *yyzz* = *yzyz* = *yzzy* = *zyyz* = *zyzy* = *zzyy* = *zxxz* = *zxzx* = *zzxx*, *xxyy* = *xyxy* = *xyyx* = *yxxy* = *yxyx* = *yyxx*, *zzzz*
C_{4v} (4*mm*), D_{2d} ($\bar{4}$2*m*), D_4 (422), D_{4h} $\left(\frac{4}{m}mm\right)$	*xxxx* = *yyyy*, *xxzz* = *xzxz* = *xzzx* = *yyzz* = *yzyz* = *yzzy* = *zyyz* = *zyzy* = *zzyy* = *zxxz* = *zxzx* = *zzxx*, *xxyy* = *xyxy* = *xyyx* = *yxxy* = *yxyx* = *yyxx*, *zzzz*
Hexagonal C_{3h} ($\bar{6}$), C_6 (6), C_{6h} $\left(\frac{6}{m}\right)$, C_{6v} (6*mm*), D_{3h} ($\bar{6}$2*m*), D_6 (622), D_{6h} $\left(\frac{6}{m}mm\right)$	*xxxx* = *yyyy* = *xxyy* + *xyxy* + *xyyx*, *xxzz* = *xzxz* = *xzzx* = *yyzz* = *yzyz* = *yzzy* = *zyyz* = *zyzy* = *zzyy* = *zxxz* = *zxzx* = *zzxx*, *xxyy* = *xyxy* = *xyyx* = *yxxy* = *yxyx* = *yyxx*, *zzzz*
Trigonal C_3 (3), C_{3i} ($\bar{3}$)	*xxxx* = *yyyy* = *xxyy* + *xyxy* + *xyyx*, *xyyz* = *xyzy* = *xzyy* = −*xxxz* = −*xxzx* = −*xzxx* = *yxyz* = *yxzy* = *yyxz* = *yyzx* = *yzxy* = *yzyx* = −*zxxx* = *zxyy* = *zyxy* = *zyyx*, *xxzz* = *xzxz* = *xzzx* = *yyzz* = *yzyz* = *yzzy* = *zyyz* = *zyzy* = *zzyy* = *zxxz* = *zxzx* = *zzxx*, *xxyy* = *xyxy* = *xyyx* = *yxxy* = *yxyx* = *yyxx*, *xxyz* = *xxzy* = *xyxz* = *xyzx* = *xzxy* = *xzyx* = −*yyyz* = −*yyzy* = −*yzyy* = *yxxz* = *yxzx* = *yzxx* = −*zyyy* = *zxxy* = *zxyx* = *zyxx*, *zzzz*
C_{3v} (3*m*), D_3 (32), D_{3d} ($\bar{3}$*m*) (mirror perpendicular to *x*) (twofold axis parallel to *x*)	*xxxx* = *yyyy* = *xxyy* + *xyxy* + *xyyx*, *xxzz* = *xzxz* = *xzzx* = *yyzz* = *yzyz* = *yzzy* = *zyyz* = *zyzy* = *zzyy* = *zxxz* = *zxzx* = *zzxx*, *xxyy* = *xyxy* = *xyyx* = *yxxy* = *yxyx* = *yyxx*, *xxyz* = *xxzy* = *xyxz* = *xyzx* = *xzxy* = *xzyx* = −*yyyz* = −*yyzy* = −*yzyy* = *yxxz* = *yxzx* = *yzxx* = −*zyyy* = *zxxy* = *zxyx* = *zyxx*, *zzzz*
Cubic *T* (23), T_h (*m*3), T_d ($\bar{4}$3*m*), *O* (432), O_h (*m*3*m*)	*xxxx* = *yyyy* = *zzzz*, *xxzz* = *xzxz* = *xzzx* = *xxyy* = *xyxy* = *xyyx* = *yyzz* = *yzyz* = *yzzy* = *yyxx* = *yxyx* = *yxxy* = *zzyy* = *zyzy* = *zyyz* = *zzxx* = *zxzx* = *zxxz*