Table 1.7.3.5. *Refractive-index conditions that determine collinear phase-matching loci in the principal planes of positive and negative biaxial crystals for three-wave SFG*

a, *b*, *c*, *d* refer to the areas given in Fig. 1.7.3.5. The types corresponding to the different DFGs are given in Table 1.7.3.1 (Fève *et al.*, 1993).

Types of SFG	Phase-matching loci in the principal planes	Inequalities determining three-wave collinear phase matching in biaxial crystals	
		Positive biaxial crystal	Negative biaxial crystal
		$n_x(\omega_i) < n_y(\omega_i) < n_z(\omega_i)$	$n_x(\omega_i) > n_y(\omega_i) > n_z(\omega_i)$
Type I	*a*	$\frac{n_{x3}}{\lambda_3} < \frac{n_{y1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2} < \frac{n_{z3}}{\lambda_3}$	$\frac{n_{x1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2} > \frac{n_{y3}}{\lambda_3} > \frac{n_{z1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2}$
	b	$\frac{n_{x1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2} < \frac{n_{y3}}{\lambda_3} < \frac{n_{z1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2}$	$\frac{n_{x3}}{\lambda_3} > \frac{n_{y1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2} > \frac{n_{z3}}{\lambda_3}$
	c	$\frac{n_{x3}}{\lambda_3} < \frac{n_{z1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2} < \frac{n_{y3}}{\lambda_3}$	$\frac{n_{x1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2} > \frac{n_{z3}}{\lambda_3} > \frac{n_{y1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2}$
	d	$\frac{n_{y1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2} < \frac{n_{x3}}{\lambda_3} < \frac{n_{z1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2}$	$\frac{n_{y3}}{\lambda_3} > \frac{n_{x1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2} > \frac{n_{z3}}{\lambda_3}$
Type II	*a*	$\frac{n_{x3}}{\lambda_3} < \frac{n_{x1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2}; \frac{n_{z1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2} < \frac{n_{z3}}{\lambda_3}$	$\frac{n_{y1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2} > \frac{n_{y3}}{\lambda_3} > \frac{n_{y1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2}$
	b	$\frac{n_{y1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2} > \frac{n_{y3}}{\lambda_3} > \frac{n_{y1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2}$	$\frac{n_{x3}}{\lambda_3} > \frac{n_{x1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2}; \frac{n_{z1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2} > \frac{n_{z3}}{\lambda_3}$
	c	$\frac{n_{x3}}{\lambda_3} < \frac{n_{x1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2}; \frac{n_{y1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2} < \frac{n_{y3}}{\lambda_3}$	$\frac{n_{z1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2} > \frac{n_{z3}}{\lambda_3} > \frac{n_{z1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2}$
	*c**	$\frac{n_{x1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2} < \frac{n_{x3}}{\lambda_3}; \frac{n_{y3}}{\lambda_3} < \frac{n_{y1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2}$	$\frac{n_{z1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2} > \frac{n_{z3}}{\lambda_3} > \frac{n_{z1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2}$
	d	$\frac{n_{x1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2} < \frac{n_{x3}}{\lambda_3} < \frac{n_{x1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2}$	$\frac{n_{y3}}{\lambda_3} > \frac{n_{y1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2}; \frac{n_{z1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2} > \frac{n_{z3}}{\lambda_3}$
	*d**	$\frac{n_{x1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2} < \frac{n_{x3}}{\lambda_3} < \frac{n_{x1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2}$	$\frac{n_{y1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2} > \frac{n_{y3}}{\lambda_3}; \frac{n_{z3}}{\lambda_3} > \frac{n_{z1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2}$
Type III	*a*	$\frac{n_{x3}}{\lambda_3} < \frac{n_{y1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2}; \frac{n_{y1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2} < \frac{n_{z3}}{\lambda_3}$	$\frac{n_{x1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2} > \frac{n_{y3}}{\lambda_3} > \frac{n_{z1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2}$
	b	$\frac{n_{x1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2} < \frac{n_{y3}}{\lambda_3} < \frac{n_{z1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2}$	$\frac{n_{x3}}{\lambda_3} > \frac{n_{y1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2}; \frac{n_{y1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2} > \frac{n_{z3}}{\lambda_3}$
	c	$\frac{n_{x3}}{\lambda_3} < \frac{n_{z1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2}; \frac{n_{z1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2} < \frac{n_{y3}}{\lambda_3}$	$\frac{n_{x1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2} > \frac{n_{z3}}{\lambda_3} > \frac{n_{y1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2}$
	*c**	$\frac{n_{z1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2} < \frac{n_{x3}}{\lambda_3}; \frac{n_{y3}}{\lambda_3} < \frac{n_{z1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2}$	$\frac{n_{x1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2} > \frac{n_{z3}}{\lambda_3} > \frac{n_{y1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2}$
	d	$\frac{n_{y1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2} < \frac{n_{x3}}{\lambda_3} < \frac{n_{z1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2}$	$\frac{n_{y3}}{\lambda_3} > \frac{n_{x1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2}; \frac{n_{x1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2} > \frac{n_{z3}}{\lambda_3}$
	*d**	$\frac{n_{y1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2} < \frac{n_{x3}}{\lambda_3} < \frac{n_{z1}}{\lambda_1} + \frac{n_{x2}}{\lambda_2}$	$\frac{n_{x1}}{\lambda_1} + \frac{n_{y2}}{\lambda_2} > \frac{n_{y3}}{\lambda_3}; \frac{n_{z3}}{\lambda_3} > \frac{n_{x1}}{\lambda_1} + \frac{n_{z2}}{\lambda_2}$
Conditions *c*, *d* are applied if		$\frac{n_{y1}}{\lambda_1} - \frac{n_{x1}}{\lambda_1}, \frac{n_{y2}}{\lambda_2} - \frac{n_{x2}}{\lambda_2} < \frac{n_{y3}}{\lambda_3} - \frac{n_{x3}}{\lambda_3}$	$\frac{n_{y1}}{\lambda_1} - \frac{n_{z1}}{\lambda_1}, \frac{n_{y2}}{\lambda_2} - \frac{n_{z2}}{\lambda_2} < \frac{n_{y3}}{\lambda_3} - \frac{n_{z3}}{\lambda_3}$
Conditions *c**, *d** are applied if		$\frac{n_{y3}}{\lambda_3} - \frac{n_{x3}}{\lambda_3} < \frac{n_{y1}}{\lambda_1} - \frac{n_{x1}}{\lambda_1}, \frac{n_{y2}}{\lambda_2} - \frac{n_{x2}}{\lambda_2}$	$\frac{n_{y3}}{\lambda_3} - \frac{n_{z3}}{\lambda_3} < \frac{n_{y1}}{\lambda_1} - \frac{n_{z1}}{\lambda_1}, \frac{n_{y2}}{\lambda_2} - \frac{n_{z2}}{\lambda_2}$

Tables 1.7.3.5 and 1.7.3.6 give, respectively, the inequalities that determine collinear phase matching in the principal planes for the three types of three-wave SFG and for the seven types of four-wave SFG.

The inequalities in Table 1.7.3.5 show that a phase-matching cone which would join the directions *a* and *d* is not possible for any type of interaction, because the corresponding inequalities have an opposite sense. It is the same for a hypothetical cone joining *b* and *c*.

The existence of type-II or type-III SFG phase matching imposes the existence of type I, because the inequalities relative to type I are always satisfied whenever type II or type III exists. However, type I can exist even if type II or type III is not allowed. A type-I phase-matched SFG in area *c* forbids phase-matching directions in area *b* for type-II and type-III SFG. The exclusion is the same between *d* and *a*. The consideration of all the possible combinations of the inequalities of Table 1.7.3.5 leads to 84 possible classes of phase-matching cones for both positive and negative biaxial crystals (Fève *et al.*, 1993; Fève, 1994). There are 14 classes for second harmonic generation (SHG) which correspond to the degenerated case ($\omega_1 = \omega_2$) (Hobden, 1967).

The coexistence of the different types of four-wave phase matching is limited as for the three-wave case: a cone joining *a* and *d* or *b* and *c* is impossible for type-I SFG. Type I in area *d*