2. BASIC CRYSTALLOGRAPHY

where $\mathbf{S}$ is a vector perpendicular to the plane reflecting the incident beam at an angle θ; the length of $\mathbf{S}$ is given by $|\mathbf{S}| = 2\sin\theta/\lambda$ [equation (2.1.4.3) in Section 2.1.4.2].

The origin of the system is chosen at the origin of the selected unit cell. Atom j is at position $\mathbf{r}_j$ with respect to the origin. Another unit cell has its origin at $t \times \mathbf{a}$, $u \times \mathbf{b}$ and $v \times \mathbf{c}$, where t, u and v are whole numbers, and $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the basis vectors of the unit cell. With respect to the first origin, its scattering is

$$F(\mathbf{S})\exp(2\pi i t\mathbf{a}\cdot\mathbf{S})\exp(2\pi i u\mathbf{b}\cdot\mathbf{S})\exp(2\pi i v\mathbf{c}\cdot\mathbf{S}).$$

The wave scattered by a crystal is the sum of the waves scattered by all unit cells. Assuming that the crystal has a very large number of unit cells ($n_1 \times n_2 \times n_3$), the amplitude of the wave scattered by the crystal is

$$W_{cr}(\mathbf{S}) = F(\mathbf{S})\sum_{t=0}^{n_1}\exp(2\pi i t\mathbf{a}\cdot\mathbf{S})\sum_{u=0}^{n_2}\exp(2\pi i u\mathbf{b}\cdot\mathbf{S})$$
$$\times \sum_{v=0}^{n_3}\exp(2\pi i v\mathbf{c}\cdot\mathbf{S}). \qquad (2.1.4.6)$$

For an infinitely large crystal, the three summations over the exponential functions are delta functions. They have the property that they are zero unless

$$\mathbf{a}\cdot\mathbf{S} = h, \quad \mathbf{b}\cdot\mathbf{S} = k \quad \text{and} \quad \mathbf{c}\cdot\mathbf{S} = l, \qquad (2.1.4.7)$$

where h, k and l are whole numbers, either positive, negative or zero. These are the Laue conditions. If they are fulfilled, all unit cells scatter in phase and *the amplitude of the wave scattered by the crystal is proportional to the amplitude of the structure factor F*. Its intensity is proportional to $|F|^2$.

$\mathbf{S}$ vectors satisfying equation (2.1.4.7) are denoted by $\mathbf{S}(hkl)$ or $\mathbf{S}(\mathbf{h})$, and the corresponding structure factors as $F(hkl)$ or $F(\mathbf{h})$.

Bragg's law for scattering by a crystal is better known than the Laue conditions:

$$2d\sin\theta = \lambda, \qquad (2.1.4.8)$$

where d is the distance between reflecting lattice planes, θ is the reflecting or glancing angle and λ is the wavelength (Fig. 2.1.4.9). It can easily be shown that the Laue conditions and Bragg's law are equivalent by combining equation (2.1.4.7) with the following information:

(1) Vector $\mathbf{S}$ is perpendicular to a reflecting plane (Section 2.1.4.2).
(2) The Laue conditions for scattering [equation (2.1.4.7)] can be written as

$$\frac{\mathbf{a}}{h}\cdot\mathbf{S} = 1; \quad \frac{\mathbf{b}}{k}\cdot\mathbf{S} = 1; \quad \frac{\mathbf{c}}{l}\cdot\mathbf{S} = 1. \qquad (2.1.4.9)$$

(3) Lattice planes always divide the unit-cell vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ into a number of equal parts (Section 2.1.1). If the lattice planes divide the $\mathbf{a}$ vector of the unit cell into h equal parts, the first index for this set of planes is h. The second index, k, is related to the division of $\mathbf{b}$ and the third index, l, to the division of $\mathbf{c}$.

From equation (2.1.4.9) it follows that vector $\mathbf{S}(hkl)$ is perpendicular to a plane determined by the points $\mathbf{a}/h$, $\mathbf{b}/k$ and $\mathbf{c}/l$, and according to conditions (3) this is a lattice plane. Therefore, scattering by a crystal can indeed be regarded as reflection by lattice planes. The projection of $\mathbf{a}/h$, $\mathbf{b}/k$ and $\mathbf{c}/l$ on vector $\mathbf{S}(hkl)$ is $1/|\mathbf{S}(hkl)|$ (Laue condition), but it is also equal to the spacing $d(hkl)$ between the lattice planes (see Fig. 2.1.1.3), and, therefore,
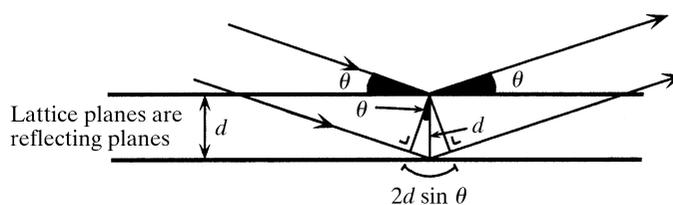

Figure 2.1.4.9
X-ray diffraction by a crystal is, in Bragg's conception, reflection by lattice planes. The beams reflected by successive planes have a path difference of $2d\sin\theta$, where d is the lattice-plane distance and θ is the reflecting angle. Positive interference occurs if $2d\sin\theta = \lambda$, 2λ, 3λ *etc.*, where λ is the X-ray wavelength. Reproduced with permission from Drenth (1999). Copyright (1999) Springer-Verlag.

$|\mathbf{S}(hkl)| = 1/d(hkl)$. Combining this with equation (2.1.4.3) yields Bragg's law, $2d\sin\theta = \lambda$ [equation (2.1.4.8)].

2.1.4.6. The structure factor

For noncentrosymmetric structures, the structure factor,

$$F(\mathbf{S}) = \sum_{j=1}^{n} f_j \exp(2\pi i\mathbf{r}_j\cdot\mathbf{S}),$$

is an imaginary quantity and can also be written as[2]

$$F(\mathbf{S}) = \sum_{j=1}^{n} f_j\cos(2\pi\mathbf{r}_j\cdot\mathbf{S}) + i\sum_{j=1}^{n} f_j\sin(2\pi\mathbf{r}_j\cdot\mathbf{S}) = A(\mathbf{S}) + iB(\mathbf{S}).$$

It is sometimes convenient to split the structure factor into its real part, $A(\mathbf{S})$, and its imaginary part, $B(\mathbf{S})$. For centrosymmetric structures, $B(\mathbf{S}) = 0$ if the origin of the structure is chosen at the centre of symmetry.

The average value of the structure-factor amplitude $|F(\mathbf{S})|$ decreases with increasing $|\mathbf{S}|$ or, because $|\mathbf{S}| = 2\sin\theta/\lambda$, with increasing reflecting angle θ.

This is caused by two factors:

(1) A stronger negative interference between the electrons in the atoms at a larger scattering angle; this is expressed in the decrease of the atomic scattering factor as a function of $\mathbf{S}$.
(2) The temperature-dependent vibrations of the atoms. Because of these vibrations, the apparent size of an atom is larger during an X-ray exposure, and the decrease in its scattering as a function of $\mathbf{S}$ is stronger. If the vibration is equally strong in all directions, it is called isotropic, and the atomic scattering factor must be multiplied by a correction factor, the temperature factor, $\exp[-B(\sin^2\theta)/\lambda^2]$. It can be shown that the parameter B is related to the mean-square displacement of the atomic vibrations, $\overline{u^2}$:

$$B = 8\pi^2\overline{u^2}.$$

In protein crystal structures determined at high resolution, each atom is given its own individual thermal parameter B.[3] Anisotropic thermal vibration is described by six parameters instead of one, and the evaluation of this anisotropic thermal vibration requires more data (X-ray intensities) than are usually available. Only at very high resolution (better than 1.5 Å) can one consider the incorporation of anisotropic temperature factors.

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[2] For convenience, we write $F(\mathbf{S})$ when we mean $F(hkl)$ or $F(\mathbf{h})$, and $\mathbf{S}$ instead of $\mathbf{S}(hkl)$ or $\mathbf{S}(\mathbf{h})$.
[3] Do not confuse the thermal parameter B with the imaginary part $B(\mathbf{S})$ of the structure factor.
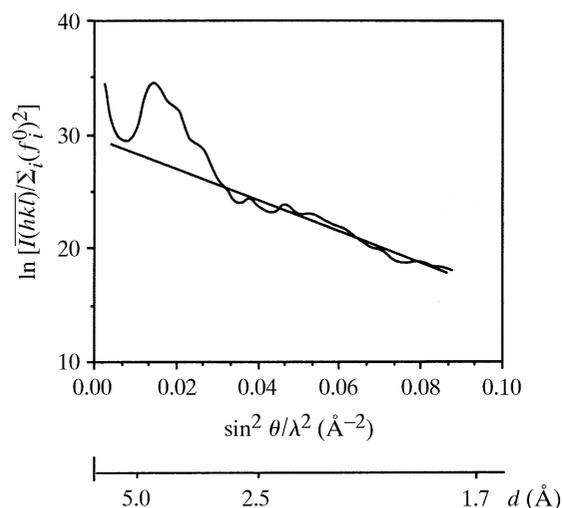
Figure 2.1.4.10
The Wilson plot for phospholipase A$_2$ with data to 1.7 Å resolution. Only beyond 3 Å resolution is it possible to fit the curve to a straight line. Reproduced with permission from Drenth (1999). Copyright (1999) Springer-Verlag.

The value of $|F(\mathbf{S})|$ can be regarded as the effective number of electrons per unit cell scattering in the direction corresponding to $\mathbf{S}$. This is true if the values of $|F(\mathbf{S})|$ are on an absolute scale; this means that the unit of scattering is the scattering by one electron in a specific direction. The experimental values of $|F(\mathbf{S})|$ are normally on an arbitrary scale. The average value of the scattered intensity, $\overline{I(\text{abs.}, \mathbf{S})}$, on an absolute scale is $\overline{I(\text{abs.}, \mathbf{S})} = \overline{|F(\mathbf{S})|^2} = \sum_i f_i^2$, where f_i is the atomic scattering factor reduced by the temperature factor. This can be understood as follows:

$$I(\text{abs.}, \mathbf{S}) = F(\mathbf{S}) \cdot F^*(\mathbf{S}) = |F(\mathbf{S})|^2$$
$$= \sum_i \sum_j f_i f_j \exp\left[2\pi i(\mathbf{r}_i - \mathbf{r}_j) \cdot \mathbf{S}\right]. \quad (2.1.4.10)$$

For a large number of reflections, $\mathbf{S}$ varies considerably, and assuming that the angles $[2\pi(\mathbf{r}_i - \mathbf{r}_j) \cdot \mathbf{S}]$ are evenly distributed over the range 0–2π for $i \neq j$, the average value for the terms with $i \neq j$ will be zero and only the terms with $i = j$ remain, giving

$$\overline{|F(\mathbf{S})|^2} = \overline{I(\text{abs.}, \mathbf{S})} = \sum_i f_i^2. \quad (2.1.4.11)$$

Because of the thermal vibrations

$$f_i^2 = \exp\left(-2B_i \sin^2\theta/\lambda^2\right)(f_i^o)^2,$$

where i denotes a specific atom and f_i^o is the scattering factor for the atom i at rest.

It is sometimes necessary to transform the intensities and the structure factors from an arbitrary to an absolute scale. Wilson (1942) proposed a method for estimating the required scale factor K and, as an additional bonus, the thermal parameter B averaged over the atoms:

$$\overline{I(\mathbf{S})} = K\overline{I(\text{abs.}, \mathbf{S})} = K \exp(-2B\sin^2\theta/\lambda^2)\sum_i (f_i^o)^2. \quad (2.1.4.12)$$

To determine K and B, equation (2.1.4.11) is written in the form

$$\ln[\overline{I(\mathbf{S})}/\sum_i (f_i^o)^2] = \ln K - 2B\sin^2\theta/\lambda^2. \quad (2.1.4.13)$$

Because f_i^o depends on $\sin\theta/\lambda$, average intensities, $\overline{I(\mathbf{S})}$, are calculated for shells of narrow $\sin\theta/\lambda$ ranges. $\ln[\overline{I(\mathbf{S})}/\sum_i(f_i^o)^2]$ is plotted against $\sin^2\theta/\lambda^2$. The result should be a straight line with slope $-2B$, intersecting the vertical axis at $\ln K$ (Fig. 2.1.4.10).

For proteins, the Wilson plot gives rather poor results because the assumption in deriving equation (2.1.4.11) that the angles, $[2\pi(\mathbf{r}_i - \mathbf{r}_j) \cdot \mathbf{S}]$, are evenly distributed over the range 0–2π for $i \neq j$ is not quite valid, especially not in the $\sin\theta/\lambda$ ranges at low resolution.

As discussed above, the average value of the structure factors, $F(\mathbf{S})$, decreases with the scattering angle because of two effects:

(1) the decrease in the atomic scattering factor f;
(2) the temperature factor.

This decrease is disturbing for statistical studies of structure-factor amplitudes. It is then an advantage to eliminate these effects by working with normalized structure factors, $E(\mathbf{S})$, defined by

$$E(\mathbf{S}) = F(\mathbf{S}) \bigg/ \left(\sum_j f_j^2\right)^{1/2}$$
$$= F(\mathbf{S}) \exp\left(B\sin^2\theta/\lambda^2\right) \bigg/ \left[\sum_j (f_j^o)^2\right]^{1/2}. \quad (2.1.4.14)$$

The application of equation (2.1.4.14) to $\overline{|E(\mathbf{S})|^2}$ gives

$$\overline{|E(\mathbf{S})|^2} = \overline{|F(\mathbf{S})|^2} \bigg/ \sum_j f_j^2 = \overline{|F(\mathbf{S})|^2} \bigg/ \overline{|F(\mathbf{S})|^2} = 1. \quad (2.1.4.15)$$

The average value, $\overline{|E(\mathbf{S})|^2}$, is equal to 1. The advantage of working with normalized structure factors is that the scaling is not important, because if equation (2.1.4.14) is written as

$$E(\mathbf{S}) = \frac{F(\mathbf{S})}{(\overline{|F(\mathbf{S})|^2})^{1/2}},$$

a scale factor affects numerator and denominator equally.

In practice, the normalized structure factors are derived from the observed data as follows:

$$E(\mathbf{S}) = F(\mathbf{S}) \exp\left(B\sin^2\theta/\lambda^2\right) \big/ \left(\varepsilon\overline{|F(\mathbf{S})|^2}\right)^{1/2}, \quad (2.1.4.16)$$

where ε is a correction factor for space-group symmetry. For general reflections it is 1, but it is greater than 1 for reflections having $\mathbf{h}$ parallel to a symmetry element. This can be understood as follows. For example, if m atoms are related by this symmetry element, $\mathbf{r}_j \cdot \mathbf{S}$ (with j from 1 to m) is the same in their contribution to the structure factor

$$F(\mathbf{h}) = \sum_{j=1}^{m} f_j \exp(2\pi i\mathbf{r}_j \cdot \mathbf{S}).$$

They act as one atom with scattering factor $m \times f$ rather than as m different atoms, each with scattering factor f. According to equation (2.1.4.11), this increases $F(\mathbf{h})$ by a factor $m^{1/2}$ on average. To make the F values of all reflections statistically comparable, $F(\mathbf{h})$ must be divided by $m^{1/2}$. For a detailed discussion, see *IT* B (2008), Chapter 2.1, by U. Shmueli and A. J. C. Wilson.

2.1.5. Reciprocal space and the Ewald sphere

A most convenient tool in X-ray crystallography is the reciprocal lattice. Unlike real or direct space, reciprocal space is imaginary. The reciprocal lattice is a superior instrument for constructing the X-ray diffraction pattern, and it will be introduced in the following way. Remember that vector $\mathbf{S}(hkl)$ is perpendicular to a reflecting plane and has a length $|\mathbf{S}(hkl)| = 2\sin\theta/\lambda = 1/d(hkl)$ (Section 2.1.4.5). This will now be applied to the boundary planes

references